# ON THE DENOMINATOR RULE AND A THEOREM BY JANOS ACZÉL 

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#### Abstract

In this note we compare and contrast arithmetic and geometric shareweighting aggregations of ratio expressions (i.e., relative scores) and we provide conditions under which they result in approximately the same value of the aggregate measure. We also provide some comparative empirical results about the differences among the suggested alternative aggregate measures.


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## On the Denominator Rule and a Theorem by Janos Aczél

In a recent paper, Färe and Karagiannis (2017) developed a rule for aggregation of ratio expressions. Examples include, technical and scale efficiency, input congestion, capacity utilization, and partial (i.e., labor) and total productivity indices. Earlier Janos Aczél (1957) also studied ratio aggregation; see also Aczél and Alsina (1986), Aczél (1990) and Roberts (1990). While Färe and Karagiannis (2017) find that under their conditions share weighted arithmetic means is the appropriate aggregation rule Aczél (1957) found that under his conditions a weighted geometric mean is the appropriate aggregation rule.

In this note we discuss and contrast the two approaches and we find conditions under which they are related to each other.

We start with the more narrow view Färe and Karagiannis (2017) took. Consider two vectors $\left(z_{1}, z_{2}\right)$ and $\left(\xi_{1}, \xi_{2}\right)$, which for example could respectively be two firms maximal and observed revenue that are used to define revenue efficiency. ${ }^{1}$ The problem they studied is:

$$
\begin{equation*}
R\left(\frac{z_{1}}{\xi_{1}}, \frac{z_{2}}{\xi_{2}}\right)=\frac{z_{1}+z_{2}}{\xi_{1}+\xi_{2}} \tag{1}
\end{equation*}
$$

and they show that aggregation of the "firm" specific relative scores $z_{1} / \xi_{1}$ and $z_{2} / \xi_{2}$ is given by a weighted arithmetic mean,

$$
\begin{equation*}
R\left(\frac{z_{1}}{\xi_{1}}, \frac{z_{2}}{\xi_{2}}\right)=\theta_{1}\left(\frac{z_{1}}{\xi_{1}}\right)+\theta_{2}\left(\frac{z_{2}}{\xi_{2}}\right), \tag{2}
\end{equation*}
$$

where $\theta_{i}=\xi_{i} /\left(\xi_{1}+\xi_{2}\right), i=1,2$. The name "denominator rule" was introduced since the weights consist of the variables $\xi_{1}, \xi_{2}$ in the denominator; see (1).

The problem Aczél (1957) studied is: ${ }^{2}$

$$
\begin{equation*}
\underline{R}\left(\frac{z_{1}}{\xi_{1}}, \frac{z_{2}}{\xi_{2}}\right)=\frac{f\left(z_{1}, z_{2}\right)}{f\left(\xi_{1}, \xi_{2}\right)} . \tag{3}
\end{equation*}
$$

He found that the most general solution to (3) is:

$$
\begin{equation*}
\underline{R}\left(\frac{z_{1}}{\xi_{1}}, \frac{z_{2}}{\xi_{2}}\right)=\left(\frac{z_{1}}{\xi_{1}}\right)^{g_{1}}\left(\frac{z_{2}}{\xi_{2}}\right)^{g_{2}} \tag{4}
\end{equation*}
$$

where $g_{1}$ and $g_{2}$ are arbitrary constants. ${ }^{3}$ This result is obtained by assuming that the general functions $\left(z_{1}, z_{2}\right) \in \mathfrak{R}_{+}^{2} \rightarrow f\left(z_{1}, z_{2}\right) \in \Re_{+}$and $\left(\xi_{1}, \xi_{2}\right) \in \mathfrak{R}_{+}^{2} \rightarrow f\left(\xi_{1}, \xi_{2}\right) \in$ $\Re_{+}$are smooth enough so that one can solve a functional equation, the Pexider, to obtain:

$$
\begin{equation*}
\underline{R}\left(\frac{z_{1}}{\xi_{1}}, \frac{z_{2}}{\xi_{2}}\right)=\frac{\prod_{i=1}^{2} z_{i}^{g_{i}}}{\prod_{i=1}^{2} \xi_{i}^{g_{i}}} \tag{5}
\end{equation*}
$$

which should be contrasted to (1) above.
The difference in the aggregation rules (2) and (4) is due to the different ways $z_{i}$ and $\xi_{i} i=1,2$ are aggregated. Färe and Karagiannis (2017) use the summations $z_{1}+z_{2} \in \Re_{+}$and $\xi_{1}+\xi_{2} \in \Re_{+}$while Aczél (1957) used a general function $\left(z_{1}, z_{2}\right) \in$ $\mathfrak{R}_{+}^{2} \rightarrow f\left(z_{1}, z_{2}\right) \in \Re_{+}$and $\left(\xi_{1}, \xi_{2}\right) \in \mathfrak{R}_{+}^{2} \rightarrow f\left(\xi_{1}, \xi_{2}\right) \in \Re_{+}$. On the other hand, the denominator rule indicates how to compute the aggregation weights while Aczél aggregation rule mentions only that they are some positive constant summing up to one. From the microeconomic theory it is known that $g_{1}$ and $g_{2}$ represent shares. For example, Domar (1961) essentially applied Aczél's rule to aggregate total factor productivity (in this case $z$ refers to output and $\xi$ to a total input index) of two completely integrated industries producing only final goods by using the share of each industry in base period's value of total output. ${ }^{4}$ In terms of our notation this corresponds to the value shares of $z_{1}$ and $z_{2}$.

A related practical issue with Aczél aggregation rule is that aggregate data is usually published in the form of totals. For this reason, aggregate performance ratios (e.g., productivity or efficiency measures) are better understood as arithmetic rather than geometric aggregates of the individual performance ratios. ${ }^{5}$ What will then be the difference in aggregate measures by using (2) instead to (4)?

In what follows we provide conditions under which the two aggregation rules result in approximately the same value of the aggregate measure. These conditions are derived following two different ways: one using a Taylor-series approximation as in Färe and Zelenyuk (2005) and another based on an approximate relation between
the geometric mean on the one hand and the arithmetic mean and the relative variance on the other hand.

Concerning the former, by taking a first-order Taylor-series approximation of (4) around $z_{1} / \xi_{1}=z_{2} / \xi_{2}=1$, namely

$$
\begin{align*}
\underline{R}\left(\frac{z_{1}}{\xi_{1}}, \frac{z_{2}}{\xi_{2}}\right) & \cong 1^{g_{1}} 1^{g_{2}}+g_{1}\left(1^{g_{1}-1} 1^{g_{2}}\right)\left(\frac{z_{1}}{\xi_{1}}-1\right)+g_{2}\left(1^{g_{1}} 1^{g_{2}-1}\right)\left(\frac{z_{2}}{\xi_{2}}-1\right) \\
& =g_{1}\left(\frac{z_{1}}{\xi_{1}}\right)+g_{2}\left(\frac{z_{2}}{\xi_{2}}\right), \tag{6}
\end{align*}
$$

one may verify that (2) and (4) are approximately equal to each other as long as

$$
\begin{equation*}
g_{i}=\theta_{i}=\frac{\xi_{i}}{\xi_{1}+\xi_{2}} \tag{7}
\end{equation*}
$$

for $i=1,2$ and $g_{1}+g_{2}=1$. That is, when the $g_{i}$ 's are set equal to the denominator rule weights, the two aggregation rules (2) and (4) result in approximately the same aggregate performance indicator. ${ }^{6}$

On the other hand, if $g_{i} \neq \theta_{i}$ for $\mathrm{i}=1,2$ and by assuming that deviations from the mean value are relatively small one can approximately relate the geometric and the arithmetic means of the relative scores as follows (Walters, 1963, p. 10):

$$
\begin{align*}
& \tilde{z} \simeq \bar{z}\left(1-\frac{1}{2} \frac{\sigma_{z}^{2}}{\bar{z}^{2}}\right)  \tag{8}\\
& \tilde{\xi} \simeq \bar{\xi}\left(1-\frac{1}{2} \frac{\sigma_{\xi}^{2}}{\bar{\xi}^{2}}\right)
\end{align*}
$$

where a bar over a variable denotes the simple arithmetic mean, i.e., $\bar{z}=\left(\frac{1}{2}\right) \sum_{i=1}^{2} z_{i}$ and $\bar{\xi}=\left(\frac{1}{2}\right) \sum_{i=1}^{2} \xi_{i}$, a tilde denotes the simple geometric mean, i.e., $\tilde{z}=\prod_{i=1}^{2} z_{i}^{1 / 2}$ and $\tilde{\xi}=\prod_{i=1}^{2} \xi_{i}^{1 / 2}$, and $\sigma^{2}$ refers to the variance. That is, the arithmetic mean gives an upward relative bias which is approximately half the relative variance, defined as the ratio of variance to the arithmetic mean. Then, using (1) and (8) one may get

$$
\begin{equation*}
R\left(\frac{z_{1}}{\xi_{1}}, \frac{z_{2}}{\xi_{2}}\right)=\frac{\bar{z}}{\bar{\xi}} \simeq \frac{\tilde{z}\left(1-\frac{1}{2} \frac{\sigma_{Z}^{2}}{\bar{z}^{2}}\right)^{-1}}{\tilde{\xi}\left(1-\frac{1}{2} \frac{\sigma_{\xi}^{2}}{\bar{\xi}^{2}}\right)^{-1}}=\left(\frac{\tilde{z}}{\tilde{\xi}}\right) \frac{\left(1-\frac{1}{2} \frac{\sigma_{\xi}^{2}}{\bar{\xi}^{2}}\right)}{\left(1-\frac{1}{2} \frac{\sigma_{Z}^{2}}{\bar{z}^{2}}\right)} \tag{9}
\end{equation*}
$$

On the other hand, we may rewrite (5) as:

$$
\begin{equation*}
\underline{R}\left(\frac{z_{1}}{\xi_{1}}, \frac{z_{2}}{\xi_{2}}\right)=\frac{\prod_{i=1}^{2} z_{i}^{1 / 2} z_{i}^{g_{i}-1 / 2}}{\prod_{i=1}^{2} \xi_{i}^{1 / 2} \xi_{i}^{g_{i}-1 / 2}}=\left(\frac{\tilde{z}}{\tilde{\xi}}\right) \prod_{i=1}^{2}\left(\frac{z_{i}}{\xi_{i}}\right)^{g_{i}-1 / 2} \tag{10}
\end{equation*}
$$

Then, (9) and (10) imply that (2) and (4) are approximately equal to each other as long as $\prod_{i=1}^{2}\left(\frac{z_{i}}{\xi_{i}}\right)^{g_{i}-1 / 2}$ is equal to $\left(1-\frac{1}{2} \frac{\sigma_{\xi}^{2}}{\bar{\xi}^{2}}\right) /\left(1-\frac{1}{2} \frac{\sigma_{Z}^{2}}{\bar{z}^{2}}\right)$. That is, when deviations from the mean value are relatively small for $z$ and $\xi$, the left-hand side of (4) is equal to the product of the simple geometric mean of relative scores $z_{1} / \xi_{1}$ and $z_{2} / \xi_{2}$ and the inverse of the adjustment factors, i.e.,

$$
\begin{equation*}
\underline{R}\left(\frac{z_{1}}{\xi_{1}}, \frac{z_{2}}{\xi_{2}}\right)=\left(\frac{\tilde{z}}{\tilde{\xi}}\right) \frac{\left(1-\frac{1}{2} \frac{\sigma_{\xi}^{2}}{\bar{\xi}^{2}}\right)}{\left(1-\frac{1}{2} \frac{\sigma_{Z}^{2}}{\bar{z}^{2}}\right)} \tag{11}
\end{equation*}
$$

and the two aggregation rules (2) and (4) results in approximately the same aggregate performance indicator. Notice however that in this case we cannot infer the value of the aggregation weights $g_{i}$ in (4). We can only approximate the value of $\underline{R}\left(\frac{z_{1}}{\xi_{1}}, \frac{z_{2}}{\xi_{2}}\right)$ as in (11).

If we further assume that $\sigma_{\xi}^{2} / \sigma_{z}^{2} \simeq \bar{\xi}^{2} / \bar{z}^{2}$, i.e., the ratio of variances of the numerator and the denominator variables are approximately equal to the ratio of their squared arithmetic means or equivalently, the coefficients of variation of the numerator and denominator variables are approximately equal, then the adjustment factors for the $z$ 's and $\xi$ 's will be about equal and in turn, (9) implies that

$$
\begin{equation*}
R\left(\frac{z_{1}}{\xi_{1}}, \frac{z_{2}}{\xi_{2}}\right) \simeq \underline{R}\left(\frac{z_{1}}{\xi_{1}}, \frac{z_{2}}{\xi_{2}}\right)=\binom{\tilde{\tilde{z}}}{\tilde{\xi}} \tag{12}
\end{equation*}
$$

That is, if deviations from mean value are relatively small and $\sigma_{\xi}^{2} / \sigma_{z}^{2} \simeq \bar{\xi}^{2} / \bar{z}^{2}$ then the left hand size of (4) is equal to the simple geometric mean of relative scores $z_{1} / \xi_{1}$ and $z_{2} / \xi_{2}$ as $g_{1}=g_{2}=1 / 2$ and the two aggregation rules (2) and (4) results in approximately the same aggregate performance indicator. Notice that in this case the aggregation weights in (2) and (4) are different.

Next we provide some comparative empirical results about the differences among the alternative aggregate scores obtained by (2), (6), (11) and (12). We do that by means of output-oriented technical efficiency scores, defined by the ratio of potential to actual output, using data from the EU Farm Accounting Data Network (FADN) for a sample of cotton farms in Greece, the size of which ranges from 426 farms in 1991 to 722 in 1995. The data include an output variable measured in terms of gross revenue and four input variables: land (measured in hectares), labor (measured in annual working hours), intermediate input expenses (for e.g., seeds, fertilizer, pesticides, maintenance, water, etc.), and capital (measured in end-of-theyear book values). Mean values of these variables are given in the upper panel of Table 1. These data are then used to estimate farm-specific output-oriented technical efficiency scores for each year separately by means of the following Data Envelopment Analysis (DEA) model, which in its envelopment form is written as:

$$
\begin{array}{ll}
\max _{\phi^{o}, \lambda_{k}^{o}} \phi^{o} & \\
\text { s.t. } & \sum_{k=1}^{K} \lambda_{k}^{o} y^{k} \geq \phi^{o} y^{o} \\
& \sum_{k=1}^{K} \lambda_{k}^{o} x_{i}^{k} \leq x_{i}^{o} \\
& \forall i=1, \ldots, I \\
& \lambda_{k}^{o} \geq 0
\end{array} \quad \forall k=1, \ldots, o, \ldots, K
$$

where $\phi$ refers to efficiency scores, $\lambda$ to intensity variables, $y$ to output quantities, and $x$ to input quantities, $i$ is used to index inputs and $k$ to index farms. Averages (arithmetic and geometric) of the technical efficiency scores are given in the middle panel of Table 1.

The estimates of the alternative aggregate efficiency scores discussed above are given in the lower panel of Tables 1. In particular, the denominator rule results are reported in the first column while the results of Aczél aggregation rule based on (6) and (11), labeled respectively as $\underline{R}^{A}$ and $\underline{R}^{B}$, are given in the next two columns. Lastly, the results of Aczél aggregation rule based on (12) are given in the last column of the middle panel of Table 1, where we report the geometric mean of the farmspecific efficiency scores. From these we can infer that $\underline{R}^{A}<\underline{R}^{B}<R<\tilde{\phi}<\bar{\phi}$, with their differences reported in the last three columns of the lower panel of Table 1. On
average, these differences are in the range of $7.4 \%$ for $R$ and $\underline{R}^{A}$, of $2.8 \%$ for $R$ and $\underline{R}^{B}$ and of $-7.2 \%$ for $R$ and $\tilde{\phi}$. In particular, using the same aggregation weights, as given in (7), the denominator and the Aczél aggregation rules result in aggregate efficiency scores that on average differ by $7.4 \%$ for the data at hand, with the former being greater than the latter. On the other hand, the denominator rule and the geometric mean of efficiency scores result in an aggregate efficiency score that on average differ by $7.2 \%$, with the latter being greater than the former. These reflect the differences in $\sigma_{\xi}^{2} / \bar{\xi}^{2}$ and $\sigma_{z}^{2} / \bar{z}^{2}$ (see the first two columns in the middle panel of Table 1), that are also depicted in the inverse of the adjustment factor that ranges between 0.93 and 0.95 (see the middle column of the middle panel of Table 1). Compared to these, (11) implies for the data at hand the smallest average difference $(2.8 \%)$ between the two aggregation schemes, with the denominator rule resulting in a higher aggregate efficiency score than Aczél aggregation rule.

Table 1: Data and results

|  | Gross <br> revenue <br> (ths $€$ ) | Land <br> (hectares) | Labor <br> (annual <br> working <br> hours) | Intermediate <br> inputs <br> (ths $€$ ) | Capital <br> (ths $€$ ) | Number <br> of farms |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 1991 | 20.83 | 11.9 | 1,904 | 6.17 | 17.11 | 426 |
| 1992 | 18.90 | 11.9 | 2,046 | 6.96 | 16.75 | 526 |
| 1993 | 26.61 | 12.0 | 2,118 | 7.27 | 15.66 | 553 |
| 1994 | 25.13 | 11.4 | 2,133 | 7.13 | 15.92 | 600 |
| 1995 | 25.23 | 11.4 | 2,132 | 7.48 | 18.37 | 722 |
|  | $\sigma_{\xi}^{2} / \bar{\xi}^{2}$ | $\sigma_{z}^{2} / \bar{z}^{2}$ | Inverse of <br> Adjustment | $\bar{\varphi}$ | $\tilde{\varphi}$ |  |
| 1991 | 0.449 | 0.338 | 0.933 | 1.907 | 1.771 |  |
| 1992 | 0.512 | 0.393 | 0.927 | 2.225 | 1.947 |  |
| 1993 | 0.454 | 0.385 | 0.955 | 1.673 | 1.588 |  |
| 1994 | 0.439 | 0.358 | 0.951 | 1.585 | 1.534 |  |
| 1995 | 0.461 | 0.362 | 0.940 | 1.603 | 1.560 |  |
|  | $R$ | $\underline{R}^{A}$ | $\underline{R}^{B}$ | $R-\underline{R}^{A}$ | $R-\underline{R}^{B}$ | $R-\tilde{\varphi}$ |
| 1991 | 1.700 | 1.602 | 1.652 | 0.098 | 0.048 | -0.071 |
| 1992 | 1.835 | 1.687 | 1.804 | 0.148 | 0.031 | -0.112 |
| 1993 | 1.527 | 1.476 | 1.516 | 0.051 | 0.011 | -0.061 |
| 1994 | 1.486 | 1.444 | 1.459 | 0.042 | 0.027 | -0.048 |
| 1995 | 1.490 | 1.458 | 1.466 | 0.032 | 0.024 | -0.070 |

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## Footnotes

${ }^{1}$ For simplicity we take the vectors in $\mathfrak{R}^{2}$ and assume that $\left(z_{1}, z_{2}\right) \geq 0$ while $\left(\xi_{1}, \xi_{2}\right)>0$. However, all the following results can easily be extended to the case of $i=1, \ldots, K$.
${ }^{2}$ See Aczél (1957, p. 150).
${ }^{3} \mathrm{He}$ explained also why $g_{i}>0, \mathrm{i}=1,2$ and $g_{1}+g_{2}=1$.
${ }^{4}$ This is referred by Domar (1961, p. 718) as the case of simple aggregation with final goods only.
${ }^{5}$ By taking this into account Massell (1961) provided an alternative to Domar (1961) aggregation scheme that contains, except for the weighted average of industry's total factor productivities, what was later known as the reallocation effect; that is, differences among industries in rates of growth and in returns to factors of production.
6 Using Cauchy's inequality, the geometric weighted average in (4) is less than or equal to the arithmetic weighted average in (2) as long as $g_{i}=\theta_{i}$ for $\mathrm{i}=1,2$.

