Partial linked-to-order delayed payment and life time effects on decaying items ordering

Ata Allah Taleizadeh

School of Industrial Engineering, College of Engineering, University of Tehran, Tehran, Iran, E-mail:Taleizadeh@ut.ac.ir

Nadia Pourmohammad-Zia

School of Industrial Engineering and Systems, Amirkabir University of Technology

E-mail:nadia.pmz@gmail.com

Ioannis Konstantaras*

Department of Business Administration, School of Business Administration, University of Macedonia, 156 Egnatia Str., Thessaloniki 54636, Greece

E-mail:ikonst@uom.gr

Abstract

Trade-credit is an influential implementation in financial transactions. This paper proposes an inventory model for decaying items with lifetime under linked-to-order partial delay in payments. More precisely, the items are gradually deteriorating and also have known maximum lifetime. The payment scheme is structured as follows: if the order quantity reaches a particular level, fully permissible trade credit is possible, otherwise the partial trade credit is offered. To the best of our knowledge, this is the first research incorporating the "Linked-to-order Trade Credit Financing" scheme for deteriorating items with lifetime. Selling price and purchasing costs are not considered equal and there is no need for the interest charged in stocks to be larger than the interest earned on investment. Theoretical results are developed to obtain the optimum solutions of the problem. The authenticity and pertinence of the model and solution procedure are illustrated through numerical results. Finally, sensitivity analysis and managerial insights are provided.

Keywords: Inventory; Partial trade credit; linked-to-order trade credit; Deterioration; Life time

Mathematics Subject Classification (2000): 90B05

1. INTRODUCTION

The competitive environment has obliged suppliers to provide a number of specified concessions for their customers in order to attract more sales (Cárdenas-Barrón and Treviño-Garza, 2014). In this regard, trade

Credit Financing (TCF) or Delayed Payment (DP) could be considered as one of the most influential concessions. In classical models, it is taken as granted that the seller is paid on receipt of the sold products (Tsao, 2010). However, the supplier can suggest a DP to the retailer in order to enhance sales. During this time, the retailer has the chance to cumulate sales revenues and earn interest on that. Trade credit patterns generally fall into two major classes: full and partial. As the name suggests, in full TCF the buyer has the opportunity to delay all of their payments. On the other hand, partial TCF allows for a fraction of the payments to be delayed and the remainder needs to be paid on purchase.

Most of the goods lose their value with the passing of time and for some of them the velocity of this process is more than the average. These items are called deteriorating or perishable products. Since deterioration imposes additional costs on the system, an appropriate inventory management of deteriorating items is of great importance. In this research, a partial linked-to-order TCF model for deteriorating items with maximum life time is being developed. Items deteriorate continuously and alongside they have their own maximum lifetime. It is assumed that, if the order quantity reaches a particular level, the fully permissible DP will be provided, otherwise the partial trade credit is offered by the supplier. This is called linked-to-order trade credit.

The remainder of the paper is organized as follows: In section 2, literature body of the problem is reviewed while in section 3 the assumptions and notations are presented. Section 4 provides the proposed model. In section 5, theoretical results are put forward. Then experimental results are provided in section 6 to prove the validity of the proposed model. Finally, section 7 finishes the paper with conclusions and recommended future research directions.

2. Literature Review

Delays in payments are a common part in mercantile transactions which can effectively enhance demand in a short-run period (Seifert et al., 2013). Accordingly, there exists rich literature in this area which falls into three main classes: classical DP models, linked-to-order DP models and DP models in the presence of deterioration, each of which will be studied briefly in what follows.

2.1. Classical delayed payment models

The model proposed by Goyal (1985) is the basis for a great number of later studies in this area. Jaggi et al. (2007) developed a two-stage DP model. They assumed that demand is dependent on the period of the DP. A similar model was studied by Huang and Hsu (2008), the solution procedure of which was not complete. Das et al.'s. (2013) research is novel. They assumed that the procurement cost of the retailer linearly depends on the delay period. Teng et al. (2014) developed an economic production quantity (EPQ) system, where TCF increased the opportunity cost, default risk and sales. Zhang et al. (2014) studied TCF as a supply chain coordination tool. In their model the producer offers discount on the condition that the buyer pays a fraction of cost before delivery time and amplifies the lot size.

2.2. Linked-to-order quantity delayed payment models

Khouja and Mehnez's (1996) research is the first attempt to model linked-to-order quantity DP. In this model, if the order quantity exceeds a certain level, DP will be offered by the supplier to the customer. Ouyang et al. (2008) proposed a model with variable manufacturing rate, price-dependent demand, linked-to-order DP. Huang (2007) extended this model by considering partial TCF. Chung et al. (2013) further extended Huang's model (2007) by relaxing the obligation for interest charged in stocks to be larger than interest earned on investment. They also proposed a different solution framework.

2.3. Delayed payments models with deterioration

Aggarwal and Jaggi (1995) generalized Goyal's (1985) work by incorporating deterioration. Liao (2008) studied an EPQ model with exponential deterioration rate and two-stage warehouse. Chung and Huang (2007) extended a double storages system for perishable items with two-stage DP. This work was extended by Liao et al. (2013) by considering that the decaying rate in rented warehouse exceeds that in owned one. Jaggi et al. (2017) developed a similar framework for deteriorating items with imperfect quality. Wang et al. (2014) proposed an inventory model for deteriorating items with lifetime under TCF which had two major impacts on their model: Increasing demand rate and also raising default risk. They also applied the proposed structure for non-deteriorating items. Wu et al. (2014) extended Wang et al. (2014) by considering two-level TCF. Down and up-stream DPs have been studied by Chen et al. (2013), Shah and Cárdenas-Barrón (2015), Banu and Mondal (2018) and Tiwari et al. (2018) for decaying items as well.

Shah (2015) introduced an inventory control model for deteriorating items where the demand is depended on the length of the credit period. They determined the optimal replenishment time and credit period under two levels of trade credit financing schemes. Annadurai and Uthayakumar (2015) set up a lot-sizing model for decaying items with stock-dependent demand and delay in payments. Dye and Yang (2015) provided a similar framework for items with general time-dependent deterioration rate under environmental considerations. Mishra et al. (2018) studied TCF for items with preservation-dependent deterioration rate. Shaikh et al. (2018) also put forth a similar inventory model for deteriorating items where shortages are admissible and the demand rate is price dependent.

Up to now, far too little attention has been paid to simultaneous study of deterioration and linked-to-order quantity DP. Chang et al. (2003) developed one of the first studies in this field. Then, Chang (2004) extended the previous model by incorporating inflation rate as well as finite planning horizon. Later, Ouyang et al. (2009) extended Chang et al. (2003) by allowing for partial TCF when the ordering quantity does not reach the specified threshold. A brief comparison of the mentioned studies is provided in Table (1).

As review of the literature reveals, TCF is an active research area which has attracted high level of academic effort. On the other hand, there are still some unfolded and overlooked directions to be surveyed. Most of the papers avoid considering deterioration as it forces more complexity on the model. Even when deterioration is not overlooked, it usually appears in its simplest form i.e. constant rate. In comparison to the classical models, linked-to-order trade credit is not deeply studied especially under partial payments. Taken all the above studies together, it could be claimed that: To the best of our knowledge this is the first academic effort which studies linked-to-order quantity TCF where partial payments are offered and the items which already have their own lifetime, deteriorate following time-dependent patterns.

3. MODEL DEVELOPMENT

The following notations are employed throughout the paper to formulate the problem:

A	Fixed order cost
D	Rate of demand per year
W	The specific threshold till which the fully trade credit is permitted
T_w	The time that W units are used

TABLE 1. REVIEW OF AFORESAID STUDIES

Reference	Trade Credit			Life time	Deterioration pattern
	Full or Partial Trade Credit	Linked to order	Two stage		
Chang et al. (2003)	Full	✓	-	-	Constant
Chang (2004)	Full	✓	-	-	Constant
Jaggi et al. (2007)	Full	-	✓	-	-
Huang (2007)	Partial	✓	-	-	-
Chung and Huang (2007	Full	-	✓	-	Constant
Huang & Hsu (2008)	Partial	-	✓	-	-
Liao (2008)	Full	-	-	-	Constant
Ouyang et al. (2009)	Partial	✓	-	-	Constant
Das et al. (2013)	Full	-	-	-	Constant
Chung et al. (2013)	Partial	✓	-	-	-
Chen et al. (2013)	Partial	-	✓	-	Constant
Liao et al. (2013)	Full	-	✓	-	Constant
Zhang et al. (2014)	Full	-	-	-	-
Wang et al. (2014)	Full	-	-	✓	Time-dependent
Wu et al. (2014)	Full	-	✓	✓	Time-dependent
Annadurai & Uthayakumar (2015)	Full	-	-	-	Constant
Shah (2015)	Full	-	✓	-	Constant
Dye & Yang (2015)	Full	-	-	-	Time-dependent
Shah & Cárdenas-Barrón (2015)	Full	✓	✓	-	Constant
Sharma (2016)	Partial	-	✓	-	-
Zhang & Lee (2017)	Full	-	-	-	Constant
Jaggi et al. (2017)	Full	-	-	-	Constant
Banu & Mondal (2018)	Full	-	✓	-	Constant
Mishra et al. (2018)	Full	-	-	-	Constant

Shaikh et al. (2018)	Full	-	-	-	Constant
Tiwari et al. (2018)	Partial	-	✓	✓	Time-dependent
Present paper	*Partial*	✓	-	✓	*Time-dependent*

T_0	Period length during which $(1-\alpha)Q$ units are used
$\theta(t)$	The time depended deterioration rate
G	The product maximum lifetime
P	Selling price
C	Purchasing cost
h	Carrying cost per unit per year
I_e	Interest which is earned
I_k	Interest which is charged
M	The DP period length
α	The permitted fraction of DPs
Q	The order quantity
T	The inventory period
$ITC(T)$	The identical costs
$ATC_i(T)$	The annual total cost of case i

Consider a situation where the retailer uses an EOQ system to manage the inventory of his/her deteriorating items. The items are under constant deterioration and have a maximum lifetime. Linked-to-order quantity type of TCF is provided from the seller to the buyer. To combat real world conditions, the selling price and purchasing cost are not equal. Moreover, there is no obligation for the interest charged in stocks (IC) to be greater than interest earned on investment (IE).

The following assumptions are applied to model the problem:

1. Demand rate is constant and shortages are not admissible.
2. The replenishment is instantaneous at an infinite rate.
3. According to Wu and Chan (2014), the deterioration rate is $\theta(t)=\frac{1}{1+G-t};\ 0\le t\le T$, where G is maximum lifetime.
4. When an order size exceeds a pre-determined quantity, full trade credit is offered. That is, if $Q>W$ (i.e. $T>T_w$), full DP is permitted and the retailer pays CQ after M time units from filling the orders. If the ordering quantity cannot reach W, the partially delayed payment will be allowed. In this case, when the order is being filed, the retailer must pay the supplier the partial payment $(1-\alpha)CQ$ and pay off the remaining balances (αCQ) at the end of the trade credit period.
5. For as long as the amount is not paid off, obtained revenues are deposited in an interest bearing account.

4. MATHEMATICAL FORMULATION

The changes in inventory level are captured by the following equation:

$$\frac{dI(t)}{dt} = -D - \theta(t)I(t); \qquad 0 \le t \le T, \quad I(T) = 0 \tag{1}$$

Solving this equation gives:

$$I(t) = D(G+1-t)\ln(\frac{G-t+1}{G-T+1}) \qquad 0 \le t \le T \tag{2}$$

Then, the order quantity is obtained as;

$$Q = I(0) = D(G+1)\ln(\frac{G+1}{G-T+1}) \tag{3}$$

From Equation (3), the time during which W units are depleted is obtained as:

$$T_w = (G+1)[1 - e^{-\frac{W}{D(G+1)}}] \tag{4}$$

If $Q > W$ ($T > T_w$), full trade credit is allowed, else the partial trade credit is offered. The buyer must pay $(1-\alpha)CQ$ and pay off αCQ at the end of DP period.

T_0 is defined as the time during which $(1-\alpha)Q$ units are depleted. From Equation (3), we have;

$$T_0 = (G+1)(1-(\frac{G+1}{G-T+1})^{\alpha-1}) \tag{5}$$

It should be noted that parameters of M, T_w, T_0 and T are those parameters that have effects on the capital cost. According to their values, seven cases are possible which are shown in Table 2.

Identical related costs are: ordering cost, holding cost and purchasing cost;

$$Ordering\ Cost: OC = \frac{A}{T} \tag{6}$$

$$Holding\ Cost: HC = \frac{h}{T}\int_0^T I(t)dt = \frac{hD}{T}\left[\frac{(1+G)^2}{2}\ln(\frac{1+G}{1+G-T}) + \frac{T^2}{4} - \frac{(1+G)T}{2}\right] \tag{7}$$

$$Purchase\ Cost: PC = \frac{CQ}{T} = \frac{CD}{T}(G+1)\ln(\frac{1+G}{1+G-T}) \tag{8}$$

Then identical cost of the system is obtained as;

$$ITC(T) = \frac{A}{T} + \frac{hD}{T}\left[\frac{(1+G)^2}{2}\ln(\frac{1+G}{1+G-T}) + \frac{T^2}{4} - \frac{(1+G)T}{2}\right] + \frac{CD}{T}(1+G)\ln(\frac{1+G}{1+G-T}) \tag{9}$$

As mentioned, because of different values of T_w and M, two groups of $M \ge T_w$ and $M < T_w$ may occur. Now for each case of the first group which is $M \ge T_w$, IC and IE will be derived.

Case 1: $T_w \le M \le T$

As depicted in Fig. 1, IC is $\frac{I_k C}{T}\int_M^T I(t)dt$ and IE is $\frac{I_e PD}{2T}M^2$. So the annual total cost function is:

$$ATC_1(T) = ITC(T) + \frac{I_k CD}{T}\left[\frac{(1+G-M)^2}{2}\ln(\frac{1+G-M}{1+G-T}) + \frac{T^2-M^2}{4} + \frac{G(T+M)+M}{2}\right] - \frac{I_e PD}{2T}M^2 \tag{10}$$

TABLE 2. POSSIBLE CASES FOR THE PERMUTATION OF THE PARAMETERS

Comparison between M and T_w	Effects of T	Effects of T_0	Cases
$M \ge T_w$	$M \le T$	_	$T_w \le M \le T$
$M \ge T_w$	$M > T$	_	$T_w \le T < M$
$M \ge T_w$	$T < T_w$	_	$T < T_w \le M$
$M < T_w$	$T_w \le T$	_	$M < T_w \le T$
$M < T_w$	$T_w > T$	$T_0 < T \le M$	$T_0 < T < M < T_w$
$M < T_w$	$T_w > T$	$T_0 < M \le T$	$T_0 < M < T < T_w$
$M < T_w$	$T_w > T$	$M < T_0 \le T$	$M < T_0 \le T < T_w$

Case 2: $T_w \le T < M$

As in Fig. 2, IC is zero and IE is $\frac{I_e PDT^2}{2T} + \frac{I_e PD}{T}(M-T)T$. So, the annual total cost is:

$$ATC_2(T) = ITC(T) - \frac{I_e PDT^2}{2T} - \frac{I_e PD}{T}(M-T)T \tag{11}$$

Case 3: $T < T_w \le M$

According to Fig. 3, IC is $\frac{I_k C}{T}\int_0^{T_0} I(t)dt + \frac{I_k C}{T}\alpha Q T_0$ and IE is $\frac{I_e PD}{2T}T^2 + \frac{I_e PD}{T}(M-T)T$. Then we have:

$$\begin{aligned} ATC_3(T) = ITC(T) &+ \frac{I_k CD(1+G)^2}{T}\left\{-\frac{1}{2}\ln(\frac{1+G}{1+G-T})\left[\alpha(\frac{1+G}{1+G-T})^{2(\alpha-1)} - 1\right] + \frac{1}{4}\left[(\frac{1+G}{1+G-T})^{2(\alpha-1)} - 1\right]\right\} \\ &- \frac{I_k CD\alpha(1+G)^2}{T}\left\{\ln(\frac{1+G}{1+G-T})\left[1-(\frac{1+G}{1+G-T})^{(\alpha-1)}\right]\right\} - \frac{I_e PD}{2T}T^2 - \frac{I_e PD}{T}(M-T)T \end{aligned} \tag{12}$$

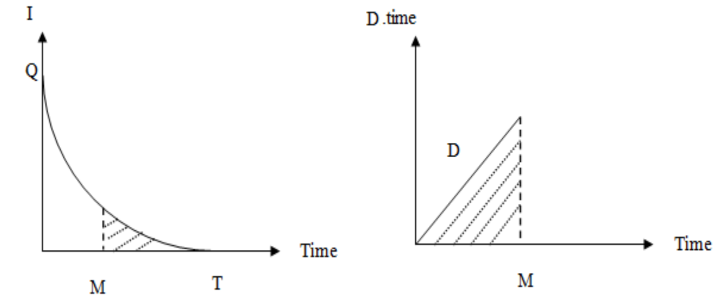

Fig. 1.The interest charged (left figure) and interest earned (right figure) for case $T_w \le M \le T$

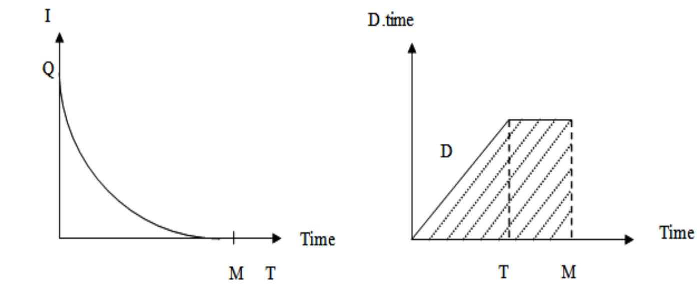

Fig. 2.The interest charged (left figure) and interest earned (right figure) for case $T_w \le T < M$

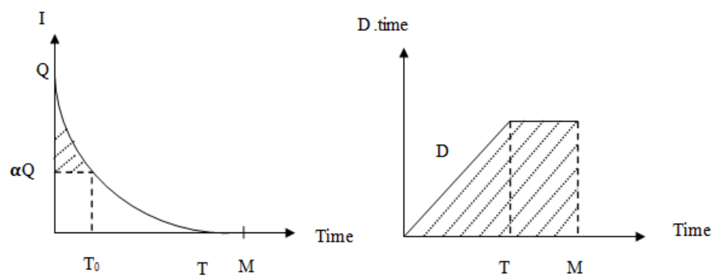

Fig. 3.The interest charged (left figure) and interest earned (right figure) for case $T < T_w \le M$

Thus, for the first group separated by $M \ge T_w$, we have:

$$ATC = \begin{cases} ATC_1 & T_w \le M \le T \\ ATC_2 & T_w \le T \le M \\ ATC_3 & T < T_w \le M \end{cases} \tag{13}$$

Case 4: $M < T_w \le T$

IC is $\frac{I_k C}{T}\int_M^T I(t)dt$ and IE is $\frac{I_e PD}{2T}M^2$. So

$$ATC_4(T) = ATC_1(T) \tag{14}$$

Case 5: $T_0 < T < M < T_w$

IC is $\frac{I_k C}{T}\int_0^{T_0} I(t)dt - \frac{I_k C}{T}\alpha Q T_0$ and IE is $\frac{I_e PD}{2T}T^2 + \frac{I_e PD}{T}(M-T)T$ .Then:

$$ATC_5(T) = ATC_3(T) \tag{15}$$

Case 6: $T_0 < M < T < T_w$

According to Fig. 4, IC is $\frac{I_k C}{T}\int_0^{T_0} I(t)dt - \frac{I_k C}{T}\alpha Q T_0 + \frac{I_k C}{T}\int_M^{T} I(t)dt$ and IE is $\frac{I_e PD}{2T}M^2$. Therefore the annual total cost is:

$$\begin{aligned} ATC_6(T) = ITC(T) &+ \frac{I_k CD(1+G)^2}{T}\left\{-\frac{1}{2}\ln(\frac{1+G}{1+G-T})\left[\alpha(\frac{1+G}{1+G-T})^{2(\alpha-1)} - 1\right] + \frac{1}{4}\left[(\frac{1+G}{1+G-T})^{2(\alpha-1)} - 1\right]\right\} \\ &+ \frac{I_k CD}{T}\left[\frac{(G-M+1)^2}{2}\ln(\frac{G-M+1}{1+G-T}) + \frac{T^2-M^2}{4} + \frac{G(T+M)+M}{2}\right] - \frac{I_e PD}{2T}M^2 \\ &- \frac{I_k CD\alpha(1+G)^2}{T}\left\{\ln(\frac{1+G}{1+G-T})\left[1-(\frac{1+G}{1+G-T})^{(\alpha-1)}\right]\right\} \end{aligned} \tag{16}$$

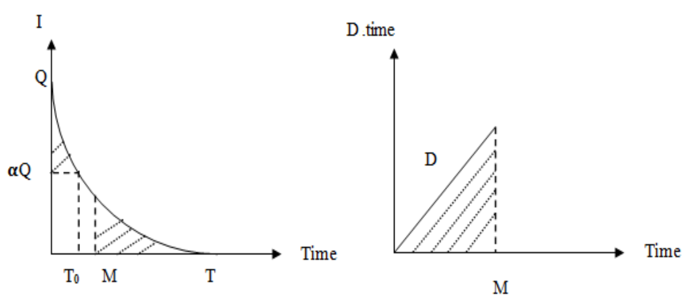

Fig. 4.The interest charged (left figure) and interest earned (right figure) for case $T_0 < M < T < T_w$

Case 7: $M < T_0 \le T < T_w$

As in Fig. 5, IC and IE are $\frac{I_k C}{T}\int_0^{T} I(t)dt - \frac{I_k C}{T}\alpha QM$ and $\frac{I_e PD}{2T}M^2$. Then

$$ATC_7 = ITC(T) + \frac{I_k CD}{T}\left[\frac{(1+G)^2}{2}\ln(\frac{1+G}{1+G-T}) + \frac{T^2}{4} - \frac{(1+G)T}{2}\right] - \frac{I_k C\alpha MD(1+G)}{T}\ln(\frac{1+G}{1+G-T}) - \frac{I_e PDM^2}{2T} \tag{17}$$

Then, the annual total costs are as below:

$$ATC = \begin{cases} ATC_1 & M < T_w \le T \\ ATC_3 & T_0 < T < M < T_w \\ ATC_6 & T_0 < M < T < T_w \\ ATC_7 & M < T_0 \le T < T_w \end{cases} \tag{18}$$

4. SOLUTION METHODOLOGY

This section seeks to obtain the optimal solution of each case, discussed above:

***Case 1**:* $T_w \le M \le T$

The necessary optimality condition for $ATC_1(T)$ gives:

$$\begin{aligned}
&-A - hD\left[\frac{(1+G)^2}{2}\ln(\frac{1+G}{1+G-T}) + \frac{T^2}{4} - \frac{(1+G)T}{2}\right] + hDT\left[\frac{(1+G)^2}{2(1+G-T)} + \frac{T}{2} - \frac{1+G}{2}\right] + CDT(\frac{1+G}{1+G-T}) \\
&-CD(1+G)\ln(\frac{1+G}{1+G-T}) + \frac{I_e PDM^2}{2} + I_k CDT\left[\frac{(1+G-M)^2}{2(1+G-T)} + \frac{T}{2} + \frac{G}{2}\right] \\
&-I_k CD\left[\frac{(1+G-M)^2}{2}\ln(\frac{1+G-M}{1+G-T}) + \frac{T^2-M^2}{4} + \frac{G(T+M)+M}{2}\right] \qquad = 0
\end{aligned} \tag{19}$$

To optimize T in $[M,\infty)$, we let:

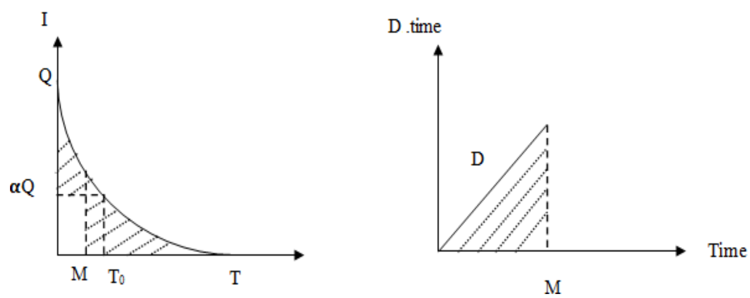

Fig. 5.The interest charged (left figure) and interest earned (right figure) for case $M < T_0 \le T < T_w$

$$\begin{aligned}
\Delta_1 &= -A - hD\left[\frac{(1+G)^2}{2}\ln(\frac{1+G}{1+G-T}) + \frac{M^2}{4} - \frac{(1+G)M}{2}\right] + hDM\left[\frac{(1+G)^2}{2(1+G-M)} + \frac{M}{2} - \frac{1+G}{2}\right] \\
&- CD(1+G)\ln(\frac{1+G}{1+G-M}) + CDM(\frac{1+G}{1+G-M}) + \frac{I_e PDM^2}{2} - I_k CD\frac{2M^2+M}{2} + I_k CDM\left[\frac{1+G-M}{2} + M\right]
\end{aligned} \tag{20}$$

Then Lemma 1 is established as:

Lemma 1.

a) If $\Delta_1 \le 0$ then there exists $T_1 \in [M,\infty)$ which is optimal, unique and satisfies Equation (19).

b) If $\Delta_1 > 0$ then $ATC_1(T)$ is optimized at $T = M$.

Proof. See Appendix A.

Case 2: $T_w \le T < M$

Similarly, $\frac{d\ ATC_2(T)}{dT} = 0$ implies:

$$-A-hD\left[\frac{(1+G)^2}{2}\ln(\frac{1+G}{1+G-T})+\frac{T^2}{4}-\frac{(1+G)T}{2}\right]+hDT\left[\frac{(1+G)^2}{2(1+G-T)}+\frac{T}{2}-\frac{1+G}{2}\right]+CDT(\frac{1+G}{1+G-T})$$
$$-CD(1+G)\ln(\frac{1+G}{1+G-T})+\frac{I_e PDT^2}{2}=0 \tag{21}$$

To show that there is an optimal T in $[T_w, M]$, we have:

$$\Delta_2=-A-hD\left[\frac{(1+G)^2}{2}\ln(\frac{1+G}{1+G-T_w})+\frac{T_w^{\,2}}{4}-\frac{(1+G)T_w}{2}\right]$$
$$+hDT_w\left[\frac{(1+G)^2}{2(1+G-T_w)}+\frac{T_w}{2}-\frac{1+G}{2}\right]+CDT_w(\frac{1+G}{1+G-T_w})-CD(1+G)\ln(\frac{1+G}{1+G-T_w})+\frac{I_e PDT_w^{\,2}}{2} \tag{22}$$

If $M\geq T_w$ then $\Delta_1\geq\Delta_2$. So lemma 2 is established as:

Lemma 2.

a) If $\Delta_2\leq 0\leq\Delta_1$ then $ATC_2(T)$ reaches its minimum at $T=T_2$ where $T_2\in[T_w, M]$ satisfies Equation (21) and is unique.

b) If $\Delta_2>0$ then $ATC_2(T)$ reaches its optimum at $T=T_w$.

c) If $\Delta_1<0$ then $ATC_2(T)$ reaches its optimum at $T=M$.

Proof. This is proven analogously to Lemma (1).

Case 3: $T<T_w\leq M$

Again, $\frac{d\ ATC_3(T)}{dT}=0$ leads to:

$$-A-hD\left[\frac{(1+G)^2}{2}\ln(\frac{1+G}{1+G-T})+\frac{T^2}{4}-\frac{(1+G)T}{2}\right]+hDT\left[\frac{(1+G)^2}{2(1+G-T)}+\frac{T}{2}-\frac{1+G}{2}\right]$$
$$+CDT(\frac{1+G}{1+G-T})-CD(1+G)\ln(\frac{1+G}{1+G-T})+\frac{I_e PDT^2}{2}$$
$$-I_k CD(1+G)^2\left\{-\frac{1}{2}\ln(\frac{1+G}{1+G-T})\left[\alpha(\frac{1+G}{1+G-T})^{2(\alpha-1)}\right]+\frac{1}{4}\left[(\frac{1+G}{1+G-T})^{2(\alpha-1)}-1\right]\right\}$$
$$+I_k CD(1+G)^2 T\left\{\begin{array}{l}-\frac{\alpha}{2}\frac{(1+G)^{2(\alpha-1)}}{(1+G-T)^{2\alpha-1}}-\frac{(\alpha-1)(1+G)^{2(\alpha-1)}}{2(1+G-T)^{2\alpha-1}}\\ -\alpha(\alpha-1)\ln(\frac{1+G}{1+G-T})\frac{(1+G)^{2(\alpha-1)}}{(1+G-T)^{2\alpha-1}}+\frac{1}{2(1+G-T)}\end{array}\right\} \tag{23}$$
$$+I_k CD\alpha(1+G)^2\left\{\ln(\frac{1+G}{1+G-T})\left[1-(\frac{1+G}{1+G-T})^{\alpha-1}\right]\right\}$$

$$-I_k CD\alpha(1+G)^2 T\left\{-\frac{(1+G)^{\alpha-1}}{(1+L-T)^{\alpha}}-(\alpha-1)\ln(\frac{1+G}{1+G-T})\frac{(1+G)^{\alpha-1}}{(1+G-T)^{\alpha}}+\frac{1}{(1+G-T)}\right\}=0$$

Δ_3 is obtained by substituting T_w for T in left side of Equation (23).

Lemma3.

a) If $\Delta_3 \geq 0$ then $ATC_3(T)$ reaches its minimum at $T_3 \in (0, T_w)$ which satisfies Equation (23) and is unique.

b) If $\Delta_3 < 0$ then there is no value for T in $(0, T_w)$ at which $ATC_3(T)$ is optimized.

Proof. See Appendix B.

Since $M \geq T_w$, we know that $\Delta_1 \geq \Delta_2$, consequently combining Lemmas (1) to (3) and considering $ATC_1(M) = ATC_2(M)$, the pursuant theorem is obtained to specify T^*.

Theorem 1. For $M \geq T_w$**,** the optimal inventory cycle (T^*) can be obtained under conditions presented in Table 3.

Proof. It can be proved under Lemmas (1) to (3) and the fact that $ATC_1(M) = ATC_2(M)$.

TABLE 3. THE OPTIMAL REPLENISHMENT CYCLE UNDER DIFFERENT CONDITIONS

Conditions	$TRC(T^*)$	T^*
$\Delta_1 \leq 0, \Delta_3 < 0$	$ATC_1(T_1)$	T_1
$\Delta_1 \leq 0, \Delta_3 \geq 0$	$Min\{ATC_1(T_1), ATC_3(T_3)\}$	T_1 or T_3
$\Delta_1 > 0, \Delta_2 \leq 0, \Delta_3 \geq 0$	$Min\{ATC_2(T_2), ATC_3(T_3)\}$	T_2 or T_3
$\Delta_1 > 0, \Delta_2 \leq 0, \Delta_3 < 0$	$ATC_2(T_2)$	T_2
$\Delta_2 > 0, \Delta_3 \geq 0$	$Min\{ATC_2(T_w), ATC_3(T_3)\}$	T_w or T_2
$\Delta_2 > 0, \Delta_3 < 0$	$ATC_2(T_w)$	T_w

Case 4: $M < T_w \leq T$

As the same approach (cases 1, 2, 3), the necessary optimality condition is $\frac{d\,ATC_1(T)}{dT} = 0$. To show uniqueness of T in $[T_w, \infty)$ at which $ATC_1(T)$ is optimized, Δ_4 is obtained by substituting T_w for T in left side of Equation (20). Consequently, we have the following lemma.

Lemma4.

a) If $\Delta_4 \leq 0$, then $ATC_1(T)$ reaches its minimum at $T_4 \in [T_w, \infty)$ which satisfies Equation (19) and is unique.

b) If $\Delta_4 > 0$, the boundary point $T = T_w$ is the optimal value.

Proof. This is proven analogously to Lemma (1).

Case 5*:* $T_0 < T < M < T_w$

Likewise, Δ_5 is obtained by substituting M for T in left side of Equation (23).

Lemma5.

a) If $\Delta_5 \geq 0$ then $ATC_3(T)$ reaches its minimum at $T_5 \in (0, M]$ which satisfies Equation (23) and is unique.

b) If $\Delta_5 < 0$, the boundary point $T = M$ is optimal.

Proof. This is proven analogously to Lemma (1).

Case 6: $T_0 < M < T < T_w$

Again, the necessary condition for $ATC_6(T)$ to reach its minimum is $\frac{d\,ATC_6(T)}{dT} = 0$ which leads to:

$$
\begin{aligned}
&-A - hD\left[\frac{(1+G)^2}{2}\ln(\frac{1+G}{1+G-T}) + \frac{T^2}{4} - \frac{(1+G)T}{2}\right] + hDT\left[\frac{(1+G)^2}{2(1+G-T)} + \frac{T-G-1}{2}\right] \\
&+CDT(\frac{1+G}{1+G-T}) - CD(1+G)\ln(\frac{1+G}{1+G-T}) \quad + \frac{I_e PDM^2}{2} \\
&-I_k CD(1+G)^2\left\{-\frac{1}{2}\ln(\frac{1+G}{1+G-T})\left[\alpha(\frac{1+G}{1+G-T})^{2(\alpha-1)}\right] + \frac{1}{4}\left[(\frac{1+G}{1+G-T})^{2(\alpha-1)} - 1\right]\right\} \\
&+I_k CD(1+G)^2 T\left\{-\frac{\alpha}{2}\frac{(1+G)^{2(\alpha-1)}}{(1+G-T)^{2\alpha-1}} + \frac{(\alpha-1)(1+G)^{2(\alpha-1)}}{2(1+G-T)^{2\alpha-1}} - \alpha(\alpha-1)\ln(\frac{1+G}{1+G-T})\frac{(1+G)^{2(\alpha-1)}}{(1+G-T)^{2\alpha-1}} + \frac{1}{2(1+G-T)}\right\} \\
&+I_k CD\alpha(1+G)^2\left\{\ln(\frac{1+G}{1+G-T})1 - (\frac{1+G}{1+G-T})^{\alpha-1}\right\} + I_k CDT\left[\frac{(1+G-M)^2}{2(1+G-T)} + \frac{T+G}{2}\right]
\end{aligned}
\tag{24}
$$

$$-I_k CD\left[\frac{(1+G-M)^2}{2}\ln(\frac{1+G-M}{1+G-T})+\frac{T^2-M^2}{4}+\frac{G(T+M)+M}{2}\right]$$

$$-I_k CD\alpha(1+G)^2 T\left\{-\frac{(1+G)^{\alpha-1}}{(1+L-T)^{\alpha}}-(\alpha-1)\ln(\frac{1+G}{1+L-T})\frac{(1+G)^{\alpha-1}}{(1+G-T)^{\alpha}}+\frac{1}{(1+G-T)}\right\}=0$$

In order to demonstrate that there exist $T\in[M,T_w)$ at which $ATC_6(T)$ is optimized; Δ_6 is obtained by substituting T_w for T in left side of Equation (24). Obviously, if $M<T_w$ then $\Delta_5<\Delta_6$ and we have lemma 6.

Lemma 6.

a) If $\Delta_5\le 0\le\Delta_6$, $ATC_6(T)$ reaches its minimum at $T_6\in[M,T_w)$ which satisfies Equation (24) and is unique.

b) If $\Delta_5>0$ then the boundary point $T=M$ is optimal.

c) If $\Delta_6<0$ then there is no value for T during $[M,T_w)$ at which $ATC_6(T)$ is minimized.

Proof. This is proven analogously to Lemma (3).

Case 7: $M<T_0\le T<T_w$

Finally $\frac{d\,ATC_7(T)}{dT}=0$ gives:

$$\begin{aligned}&-A-hD\left[\frac{(1+G)^2}{2}\ln(\frac{1+G}{1+G-T})+\frac{T^2}{4}-\frac{(1+G)T}{2}\right]+hDT\left[\frac{(1+G)^2}{2(1+G-T)}+\frac{T-G-1}{2}\right]\\&+CDT(\frac{1+G}{1+G-T})-CD(1+G)\ln(\frac{1+G}{1+G-T})\ +\frac{I_e PDM^2}{2}+I_k CDT\left[\frac{(1+G)^2}{21+G-T}+\frac{T-(1+G)}{2}\right]-\frac{I_k C\alpha MD(1+G)}{1+G-T}T\\&-I_k CD\left[\frac{(1+G)^2}{2}\ln(\frac{1+G}{1+G-T})+\frac{T^2}{4}-\frac{(1+G)T}{2}\right]+I_k C\alpha MD(1+G)\ln(\frac{1+G}{1+G-T})=0\end{aligned}\qquad(25)$$

In order to demonstrate that there exists $T\in(M,T_w)$ at which $ATC_7(T)$ is minimized, Δ_7 and Δ_8 are obtained by substituting M and T_w for T in left side of Equation (25) respectively. Then these are used in Lemma 7.

Lemma 7.

a) If $\Delta_7\le 0\le\Delta_8$ then $ATC_7(T)$ reaches its minimum $T=T_7$ where $T_7\in(M,T_w)$, satisfies Equation (25) and is unique.

b) If $\Delta_7>0$ then there is no value for T_7 in (M,T_w) at which $ATC_7(T)$ is optimized.

c) If $\Delta_8<0$ then there is no value for T_7 in (M,T_w) at which $ATC_7(T)$ is optimized.

Proof. This is proven analogously to Lemma (3).

If $M < T_w$, we have $\Delta_5 < \Delta_6$ and $\Delta_7 < \Delta_8$. Then combining Lemmas (4) to (7) and considering $ATC_3(M) = ATC_6(M)$, the following theorem is obtained to determine T^* for cases 4, 5, 6 and7.

Theorem 2. For $M < T_w$ the optimal inventory period can be obtained under conditions presented in Table 4.

Proof. It can be proved under Lemmas (4) to (7) and the fact that $ATC_3(M) = ATC_6(M)$.

Regarding to the possible intervals of period length in the sixth and the seventh cases, represented in section 4, we should note that,

- After calculating T_0 by Equation (5) with respect to T_6, if $M < T_0$ then T_6 does not have a feasible value and its related cost is omitted from our calculations and comparisons.
- After calculating T_0 by Equation (5) with respects to T_7, if $M > T_0$ then T_7 does not have a feasible value and its related cost is omitted from our calculations and comparisons.

By using Theorems (1) and (2), the following algorithm procedure is developed to solve the problem at hand. Moreover its flowchart is presented in Fig. 6.

Algorithm:

Step 1- Compare the values of M and T_w, if $M \geq T_w$ go to step 2, otherwise go to step 3.

Step 2- Calculate Δ_1, Δ_2 and Δ_3 respectively. Then (applying Theorem (1)), use Table 3 to obtain the optimal solution.

Step 3- Calculate Δ_4, Δ_5, Δ_6, Δ_7 and Δ_8 respectively. Then (applying Theorem (2)), use Table 4 to obtain the optimal solution.

Step 4-End

TABLE 4
THE OPTIMAL REPLENISHMENT CYCLE UNDER DIFFERENT CONDITIONS

Conditions	$TRC(T^*)$	T^*
$\Delta_4 \leq 0, \Delta_5 \geq 0, \Delta_7 \leq 0, \Delta_8 \geq 0$	$Min\{ATC_4(T_4), ATC_5(T_5), ATC_7(T_7)\}$	T_4 or T_5 or T_7
$\Delta_4 \leq 0, \Delta_5 \geq 0, \Delta_7 > 0$	$Min\{ATC_4(T_4), ATC_5(T_5)\}$	T_5 or T_4
$\Delta_4 \leq 0, \Delta_5 \geq 0, \Delta_8 < 0$	$Min\{ATC_4(T_4), ATC_5(T_5)\}$	T_5 or T_4
$\Delta_4 \leq 0, \Delta_5 < 0, \Delta_6 \geq 0, \Delta_7 \leq 0, \Delta_8 \geq 0$	$Min\{ATC_4(T_4), ATC_6(T_6), ATC_7(T_7)\}$	T_4 or T_6 or T_7
$\Delta_4 \leq 0, \Delta_5 < 0, \Delta_6 \geq 0, \Delta_7 > 0$	$Min\{ATC_4(T_4), ATC_6(T_6)\}$	T_6 or T_4
$\Delta_4 \leq 0, \Delta_5 < 0, \Delta_6 \geq 0, \Delta_8 < 0$	$Min\{ATC_4(T_4), ATC_6(T_6)\}$	T_6 or T_4
$\Delta_4 \leq 0, \Delta_6 < 0, \Delta_7 \leq 0, \Delta_8 \geq 0$	$Min\{ATC_4(T_4), ATC_7(T_7)\}$	T_7 or T_4

$\Delta_4 \le 0,\ \Delta_6 < 0, \Delta_7 > 0$	$ATC_4(T_4)$	T_4
$\Delta_4 \le 0, \Delta_6 < 0, \Delta_8 < 0$	$ATC_4(T_4)$	T_4
$\Delta_4 > 0, \Delta_5 \ge 0, \Delta_7 \le 0, \Delta_8 \ge 0$	$Min\{ATC_4(T_w), ATC_5(T_5), ATC_7(T_7)\}$	T_w or T_5 or T_7
$\Delta_4 > 0, \Delta_5 \ge 0, \Delta_7 > 0$	$Min\{ATC_4(T_w), ATC_5(T_5)\}$	T_5 or T_w
$\Delta_4 > 0, \Delta_5 \ge 0,\ \Delta_8 < 0$	$Min\{ATC_4(T_w), ATC_5(T_5)\}$	T_5 or T_w
$\Delta_4 > 0, \Delta_5 < 0, \Delta_6 \ge 0, \Delta_7 \le 0, \Delta_8 \ge 0$	$Min\{ATC_4(T_w), ATC_6(T_6), ATC_7(T_7)\}$	T_w or T_6 or T_7
$\Delta_4 > 0, \Delta_5 < 0, \Delta_6 \ge 0, \Delta_7 > 0$	$Min\{ATC_4(T_w), ATC_6(T_6)\}$	T_6 or T_w
$\Delta_4 > 0, \Delta_5 < 0, \Delta_6 \ge 0, \Delta_8 < 0$	$Min\{ATC_4(T_w), ATC_6(T_6)\}$	T_6 or T_w
$\Delta_4 > 0, \Delta_6 < 0, \Delta_7 \le 0, \Delta_8 \ge 0$	$Min\{ATC_4(T_w), ATC_7(T_7)\}$	T_7 or T_w
$\Delta_4 > 0,\ \Delta_6 < 0, \Delta_7 > 0$	$ATC_4(T_w)$	T_w
$\Delta_4 > 0, \Delta_6 < 0, \Delta_8 < 0$	$ATC_4(T_w)$	T_w

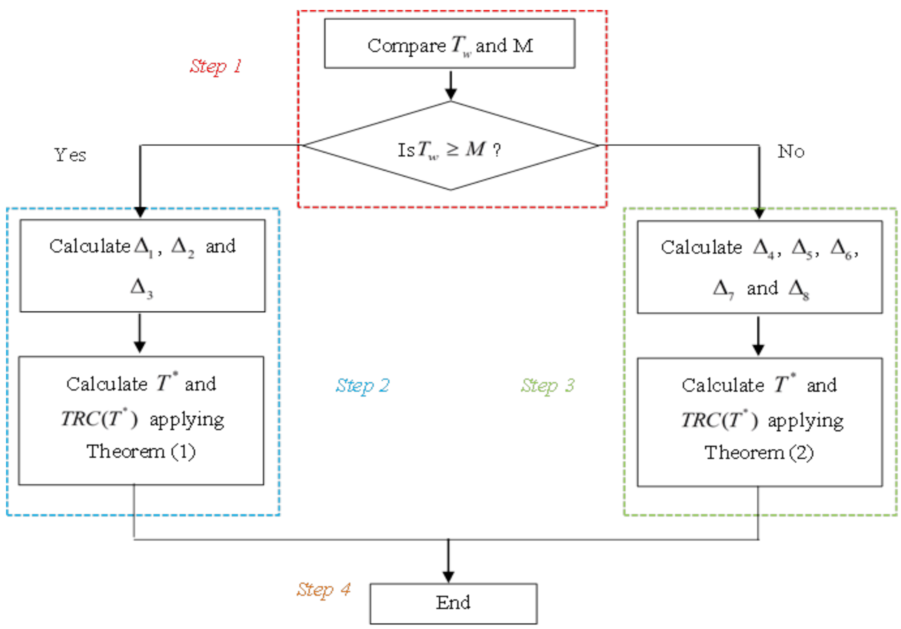

Fig. 6. Flowchart of the proposed algorithm

5. Numerical Results

Example. Assume $h = \$5 / unit / year$ $A = \$50 / order$, $I_k = 0.1\ \$ / year$, $M = 0.2\ year$, $D = 1000\ units / year$, $I_p = 0.07\ \$ / year$. For different inputs $W = (100, 200, 300)$, $\alpha = (0.2, 0.5, 0.8)$ and $C = (10, 20, 30)$, the optimal replenishment cycle is calculated and displayed in Table 5.

TABLE 5. Numerical Results

α	W	C	T^*	Q^*	$ATC^*(T^*)$
		10	$T_1 = 0.2312$	245.697	12113.15

	100	20	$T_1 = 0.2256$	239.369	13992.227
		30	$T_1 = 0.2143$	226.673	16723.761
		10	$T_w = 0.1903$	200	19308.35
0.2	200	20	$T_w = 0.1903$	200	19308.35
		30	$T_2 = 0.1846$	193.682	20981.251
		10	$T_5 = 0.1793$	187.852	22371.232
	300	20	$T_5 = 0.1697$	177.335	23520.314
		30	$T_5 = 0.1614$	168.285	24571.021
		10	$T_1 = 0.2312$	245.697	12113.15
	100	20	$T_1 = 0.2256$	239.369	13992.227
		30	$T_1 = 0.2143$	226.673	16723.761
		10	$T_w = 0.1903$	200	19308.35
0.5	200	20	$T_w = 0.1903$	200	19308.35
		30	$T_2 = 0.1874$	196.769	20721.063
		10	$T_5 = 0.1832$	192.141	20464.203
	300	20	$T_5 = 0.1788$	187.303	21324.166
		30	$T_5 = 0.1675$	174.932	22761.031
		10	$T_1 = 0.2312$	245.697	12113.15
	100	20	$T_1 = 0.2256$	239.369	13992.227
		30	$T_1 = 0.2143$	226.673	16723.761
		10	$T_w = 0.1903$	200	19308.35
0.8	200	20	$T_w = 0.1903$	200	19308.35
		30	$T_2 = 0.1893$	199.055	20410.122
		10	$T_5 = 0.1940$	204.065	20316.452
	300	20	$T_5 = 0.1878$	197.211	20211.901
		30	$T_5 = 0.1851$	194.233	21075.607

According to the provided results, the sequent managerial insights are obtained:

1. In partially DP case, for a fixed value of W and C, by increasing α significant increase in the value of Q^* and decrease in $ATC^*(T^*)$ is observed. For instance, for $W = 300$ and $C = 10$, increasing α from 0.2 to 0.5 results in a 2.5% increase in the value of Q^* and 8.5% decrease in $ATC^*(T^*)$. So if the buyer has the choice among different suppliers, they could effectively manage the costs by selecting the one who offers a higher partial TCF fraction.

2. For fixed α and C, raising W decreases Q^* and increases the value of $ATC^*(T^*)$. For example, when $\alpha = 0.2$, $C = 10$ and W increases from 100 to 200, Q^* decreases 18.5% and $ATC^*(T^*)$ 59%. This implies that the retailer cannot reach the quantity threshold for fully DP. Then, by increasing *W,* a drop in order quantity is observed and the retailer's costs increase drastically.
3. For a fixed value of α and *W*, raising the value of *C* will decrease the value of Q^* and will increase the value of $ATC^*(T^*)$. For instance, when $W = 100$ and $\alpha = 0.5$ increasing C from 10 to 20, results in a 2.5% reduction in the order size and 15% increase in the annual total cost. This is normal as we observe in classic EOQ model. That is the changes in purchasing cost have direct relation with annual total cost and reverse connection with ordering quantity.

As identical cost of the system entails ordering cost as well as inventory holding and purchasing cost, it is promising to analyze the impact of different values of inputs A, h and C on T^*, Q^* and $ATC^*(T^*)$. The related findings are shown in Table 6.

TABLE 6. SENSITIVITY ANALYSIS OF IDENTICAL COST PARAMETERS

Parameter	Changes (%)	Change in (%)		
		T	Q	$ATC^*(T^*)$
C	-50%	+52.83%	+117.42%	-122.43%
	-25%	+33.49%	+68.53%	-71.55%
	+25%	-34.11%	-68.95%	+71.41%
	+50%	-53.25%	-118.17%	+122.39%
h	-50%	+21.52%	+21.02%	-2.93%
	-25%	+12.48%	+11.98%	-1.65%
	+25%	-12.73%	-12.11%	+1.94%
	+50%	-21.52%	-21.28%	+3.15%
A	-50%	-22.08%	-21.03%	-3.45%
	-25%	-13.59%	-12.64%	-1.82%
	+25%	+13.93%	+12.85%	+2.05%
	+50%	+22.11%	+21.39%	+3.94%

According to the results presented in Table 6, the following insights can be obtained.

1. The optimal replenishment period (T^*) decreases with raising the value of h and C. It also increases with any increase in the value of A. T^* is mildly sensitive to the changes in h and A, while it is extremely sensitive to the variations in C.

2. The optimal Q^* decreases with a rise in h and C. It increases as the value of A rises. T^* is relatively sensitive to the changes in h and A, whereas it is extremely sensitive to the variations in C. This is identical to changes in the replenishment cycle.
3. The optimal $ATC^*(T^*)$ decreases with reduction in h, C and A. The impact of changes in C is much more notable than h or A. This suggests a very tricky fact to the retailer: if the retailer has the choice among different suppliers, it is logical to prefer the one with lowest purchasing cost over other cost elements.

6. CONCLUSION

In this paper, an EOQ model with the linked-to-order TCF and under deterioration was developed. We assumed that deterioration rate is time-dependent and product has maximum lifetime. In our model, if the order quantity reaches a certain level the fully DP will be possible, otherwise partial DP is offered from the supplier. Several Lemmas and Theorems were developed to solve the model. The authenticity and pertinence of the model and solution procedure were illustrated by experimental results.

Our findings reveal that as the value of the fraction of the DP increases, the optimum order size rises and reduction occurs in annual total cost. Moreover, when the unit purchasing cost increases the order quantity and the annual total cost decreases and increases respectively.

Current work can be extended in some directions: Firstly, other time-dependent deterioration patterns such Weibull might be applied. Secondly, shortages can be admissible in the model either as lost sales or backorders. Then revenue management policies can be embedded to the model by developing a price-dependent and (or) promotion-dependent demand function. Considering two-level TCF is another possible research direction. Finally, adding quantity discounts and time value of the money may provide a more practical structure. Also we suggest incorporating the partial linked to order delayed payment to the works of Taleizadeh et al (2013, 2015), Taleizadeh and Noori (2015) and Tat et al. (2015).

References

Aggarwal, S. P., Jaggi, C. K., 1995. Ordering policies of deteriorating items under permissible delay in payments. Journal of Operational Research Society 46, 658-662.

Annadurai, K., Uthayakumar, R., 2015. Decaying inventory model with stock-dependent demand and shortages under two-level trade credit International Journal of Advanced Manufacturing Technology 77, 525-543.

Banu, A., Mondal, S.K., 2018. Analyzing an inventory model with two-level trade credit period including the effect of customers' credit on the demand function using q-fuzzy number. Operational Research, 1-29.

Cárdenas-Barrón, L.E., Treviño-Garza, G., 2014. An optimal solution to a three echelon supply chain network with multi-product and multi-period. Applied Mathematical Modelling 38, 1911-1918.

Chang, C. T., 2004. An EOQ model with deteriorating items under inflation when supplier credits linked to order quantity. International Journal of Production Economics 88, 307-316.

Chang, C. T., Ouyang, L. Y., Teng, J. T., 2003. An EOQ model for deteriorating items under supplier credits linked to ordering quantity. Applied Mathematical Modeling 27, 983-996.

Chen, S. C., Teng, J. T, Skouri, K., 2013. Economic production quantity models for deteriorating items with up-stream full trade credit and down-stream partial trade credit. International Journal of Production Economics, 155, 302-309.

Chung, K. J., Huang, T. S., 2007. The optimal retailer's ordering policies for deteriorating items with limited storage capacity under trade credit financing. International Journal of Production Economics 106, 127-145.

Chung, K. J., Lin, S. D., Srivastava, H. M., 2013. The inventory models under conditional trade credit in a supply chain system. Applied Mathematical Modeling 37, 10036-10052.

Das, B. C., Das, B., & Mondal, S. K., 2013. Integrated supply chain model for a deteriorating item with procurement cost dependant credit period. Computers and Industrial Engineering, 64, 788-796.

Dave, U., 1985. Letters and viewpoints on Economic order quantity under conditions of permissible delay in payments. Journal of Operational Research Society 36, 1069-1070.

Dye, C.Y., Yang, C.T., 2015.Sustainable trade credit and replenishment decisions with credit-linked demand under carbon emission constraints. European Journal of Operational Research 244, 187-200.

Giri, B.C., Sharma, S., 2016. Optimal ordering policy for an inventory system with linearly increasing demand and allowable shortages under two levels trade credit financing. Operational Research 16, 25-50.

Goyal, S. K. 1985. Economic order quantity under conditions of permissible delay in payments. Journal of Operational Research Society 35, 335-338.

Huang, Y. F., 2007. Economic order quantity under conditionally permissible delay in payments. European Journal of Operational Research 176, 911-924.

Huang, Y. F., Hsu, K. H., 2008. An EOQ model under retailer partial trade credit policy in supply chain. International Journal of Production Economics 112, 655-664.

Jaggi, C. K., Aggarwal, K. K., Goel, S. K., 2007. Retailer’s optimal ordering policy under two stage TCF. Advanced Modeling Optimization 9, 67-80.

Jaggi, C.K., Cárdenas-Barrón, L.E., Tiwari, S., Shafi, A.A., 2017. Two-warehouse inventory model for deteriorating items with imperfect quality under the conditions of permissible delay in payments. Scientia Iranica. Transaction E, Industrial Engineering 24,390.

Jamal, A. M. M., Sarker, B. R, Wang, S., 1997. An ordering policy for deteriorating items with allowable shortage and permissible delay in payment. Journal of Operational Research Society 48, 826-833.

Khouja, M., Mehrez, A., 1996. Optimal inventory policy under different supplier credit policies. Journal of Manufacturing Systems 15, 334-339.

Liao, J. J., 2008. An EOQ model with non-instantaneous receipt and exponentially deteriorating items under two-level trade credit. International Journal of Production Economics 113, 852-861.

Liao, J. J., Chung, K. J., Huang, K. N., 2013. A deterministic inventory model for deteriorating items with two warehouses and trade credit in a supply chain system. International Journal of Production Economics 146, 557-565.

Mishra, U., Tijerina-Aguilera, J., Tiwari, S., Cárdenas-Barrón, L.E., 2018. Retailer's Joint Ordering, Pricing, and Preservation Technology Investment Policies for a Deteriorating Item under Permissible Delay in Payments.Mathematical Problems in Engineering, 2018.

Ouyang, L. Y, Ho, C. H., Su, C. H., 2008. Optimal strategy for an integrated system with variable production rate when the freight rate and trade credit are both linked to the order quantity. International Journal of Production Economics 115, 151-162.

Ouyang, L. Y., Teng, J. T., Goyal, S. K., & Yang, C. T. (2009). An economic order quantity model for deteriorating items with partially permissible delay in payments linked to order quantity. *European Journal of Operational Research*, *194*(2), 418-431.

Seifert, D., Seifert, R.W., Protopappa-Sieke, M., 2013. A review of trade credit literature: Opportunities for research in operations. European Journal of Operational Research 231, 245-256.

Shah, N.H., 2015. Retailer's replenishment and credit policies for deteriorating inventory under credit period-dependent demand and bad-debt loss, TOP, 23(1), 298-312.

Shah, N.H., Cárdenas-Barrón, L.E., 2015.Retailer's decision for ordering and credit policies for deteriorating items when a supplier offers order-linked credit period or cash discount. Applied Mathematics and Computation 259,569-578.

Shaikh, A.A., Bhunia, A.K., Cárdenas-Barrón, L.E., Sahoo, L, Tiwari, S., 2018. A Fuzzy Inventory Model for a Deteriorating Item with Variable Demand, Permissible Delay in Payments and Partial Backlogging with Shortage Follows Inventory (SFI) Policy. International Journal of Fuzzy Systems 20, 1606-1623.

Taleizadeh A.A., Wee H.M., Jolai F. 2013. Revisiting fuzzy rough economic order quantity model for deteriorating items considering quantity discount and prepayment. Mathematical and Computer Modeling, 57 (5-6): 1466-1479.

Taleizadeh A.A., Kalantary S.S., Cárdenas-Barrón L.E., 2015a. Determining optimal price, replenishment lot size and number of shipment for an EPQ model with rework and multiple shipments, Journal of Industrial and Management Optimization, 11(4), 1059-1071.

Taleizadeh A.A., Noori-Daryan M., 2015. Pricing, Manufacturing and Inventory Policies for Raw Material in a Three-Level Supply Chain, International Journal of Systems Science, 47(4): 919-931.

Tat R., Taleizadeh A.A., Esmaeili M., 2015. Developing EOQ model with non-instantaneous deteriorating items in vendor-managed inventory (VMI) system. International Journal of Systems Science, 46(7), 1257-1268.

Teng, J. T., Lou, K. R., Wang, L. 2014. Optimal trade credit and lot size policies in economic production quantity models with learning curve production costs. International Journal of Production Economics, 155, 318-323.

Tiwari, S., Cárdenas-Barrón, L.E., Goh, M., Shaikh, A.A., 2018.Joint pricing and inventory model for deteriorating items with expiration dates and partial backlogging under two-level partial trade credits in supply chain. International Journal of Production Economics 200, 16-36.

Tsao, Y. C., 2010. Two-phase pricing and inventory management for deteriorating and fashion goods under trade credit. Mathematical Methods of Operations Research 72, 107-127.

Ventura, E., William, S. T., Smith, V. L., Machol, R. E., Goyal, S. K., 1985. Letters and viewpoints. Journal of Operational Research Society 36, 965-967.

Wang, W.C., Teng, J.T., Lou, K.R., 2014. Seller's optimal credit period and cycle time in a supply chain for deteriorating items with maximum lifetime. European Journal of Operational Research 232, 315-321.

Wu, J., Chan, Y. L., 2014. Lot-sizing policies for deteriorating items with expiration dates and partial trade credit to credit-risk customers. International Journal of Production Economics, 155, 292-301.

Wu, J., Ouyang, L.Y., Cárdenas-Barrón, L.E., Goyal, S.K., 2014. Optimal credit period and lot size for deteriorating items with expiration dates under two-level trade credit financing. European Journal of Operational Research 237, 898-908.

Zhang, Q., Dong, M., Luo, J., Segerstedt, A., 2014. Supply chain coordination with trade credit and quantity discount incorporating default risk. International Journal of Production Economics, 152, 352-360.

Taleizadeh A.A., Wee H.M., Jolai F. 2013. Revisiting fuzzy rough economic order quantity model for deteriorating items considering quantity discount and prepayment. Mathematical and Computer Modeling, 57 (5-6): 1466-1479.

Taleizadeh A.A., Kalantary S.S., Cárdenas-Barrón L.E., 2015. Determining optimal price, replenishment lot size and number of shipment for an EPQ model with rework and multiple shipments, Journal of Industrial and Management Optimization, 11(4), 1059-1071.

Tat R., Taleizadeh A.A., Esmaeili M., 2015. Developing EOQ model with non-instantaneous deteriorating items in vendor-managed inventory (VMI) system. International Journal of Systems Science, 46(7), 1257-1268.

Taleizadeh A.A., Noori-Daryan M., 2015. Pricing, Manufacturing and Inventory Policies for Raw Material in a Three-Level Supply Chain, International Journal of Systems Science, 47(4): 919-931.

APPENDIX A.

In order to prove lemma (1) set

$$F_1(T) = -A - hD\left[\frac{(1+G)^2}{2}\ln(\frac{1+G}{1+G-T}) + \frac{T^2}{4} - \frac{(1+G)T}{2}\right] + \frac{I_e PDM^2}{2} + hDT\left[\frac{(1+G)^2}{2(1+G-T)} + \frac{T}{2} - \frac{1+G}{2}\right]$$
$$+ CDT(\frac{1+G}{1+G-T}) \quad - CD(1+G)\ln(\frac{1+G}{1+G-T}) + I_k CDT\left[\frac{(1+G-M)^2}{2(1+G-T)} + \frac{T+M}{2}\right] \qquad \text{(A1)}$$
$$- I_k CD\left[\frac{(1+G-M)^2}{2}\ln(\frac{1+G-M}{1+G-T}) + \frac{T^2-M^2}{4} + \frac{M(T+M)+M}{2}\right]$$

$\frac{dF_1(T)}{dT}$ with respect to $T \in [M,\infty)$ yields:

$$\frac{dF_1(T)}{dT} = hDT\left[\frac{(1+G)^2}{2(1+G-T)^2} + \frac{1}{2}\right] + CD\frac{1+G}{(1+G-T)^2} + I_k CDT\left[\frac{(1+G-M)^2}{2(1+G-T)^2} + \frac{1}{2}\right] > 0 \qquad \text{(A2)}$$

Therefore, $F_1(T)$ is strictly increasing function of $T \in [M,\infty)$. From Equation (A1), $F_1(M) = \Delta_1$ and $\lim_{T\to\infty} F_1(T) = \infty$. Thereafter, by using the Intermediate Value Theorem, we could claim that there exist a unique value of $T \in [M,\infty)$, say T_1, such that $F_1(T_1) = 0$. Furthermore:

$$\frac{d^2TRC_1(T)}{dT^2}\Big|_{T=T_1} = \frac{D}{T_1}\left\{\frac{(h+C)(1+G)^2}{2(1+G-T_1)} + \frac{h}{2} + I_k C\left[\frac{(1+G-M)^2}{2(1+G-T_1)^2} + \frac{1}{2}\right]\right\} > 0 \qquad \text{(A3)}$$

Then, apparently $T_1 \in [M,\infty)$ is a unique optimal value for $TRC_1(T)$.

If $\Delta_1 > 0$, then for all $T \in [M,\infty)$, $F_1(T) > 0$ and $\frac{dTRC_1(T)}{dT} = \frac{F_1(T)}{T^2} > 0$. Consequently, $TRC_1(T)$ is a strictly increasing function of T in the interval $T \in [M,\infty)$ .So, in this case, $TRC_1(T)$ has an optimal solution at $T = M$ and this completes the proof.

APPENDIX B.

In order to prove Lemma (3) set;

$$F_3(T) = -A - hD\left[\frac{(1+G)^2}{2}\ln(\frac{1+G}{1+G-T}) + \frac{T^2}{4} - \frac{(1+G)T}{2}\right] + hDT\left[\frac{(1+G)^2}{2(1+G-T)} + \frac{T}{2} - \frac{1+G}{2}\right] - CD(1+G)\ln(\frac{1+G}{1+G-T})$$
$$+CDT(\frac{1+G}{1+G-T}) + \frac{I_e PDT^2}{2} \quad -I_k CD(1+G)^2\left\{-\frac{1}{2}\ln(\frac{1+G}{1+G-T})\left[\alpha(\frac{1+G}{1+G-T})^{2(\alpha-1)}\right] + \frac{1}{4}\left[(\frac{1+G}{1+G-T})^{2(\alpha-1)} - 1\right]\right\} \qquad \text{(B1)}$$

$$+I_k CD(1+G)^2 T\left\{-\frac{\alpha}{2}\frac{(1+G)^{2(\alpha-1)}}{(1+G-T)^{2\alpha-1}}-\frac{\alpha-1}{2}\frac{(1+G)^{2(\alpha-1)}}{(1+G-T)^{2\alpha-1}}-\alpha(\alpha-1)\ln(\frac{1+G}{1+G-T})\frac{(1+G)^{2(\alpha-1)}}{(1+G-T)^{2\alpha-1}}+\frac{1}{2(1+G-T)}\right\}$$

$$+I_k CD\alpha(1+G)^2\left\{\ln(\frac{1+G}{1+G-T})\left[1-(\frac{1+G}{1+G-T})^{\alpha-1}\right]\right\}$$

$$-I_k CD\alpha(1+G)^2 T\left\{-\frac{(1+G)^{\alpha-1}}{(1+G-T)^{\alpha}}-\frac{\alpha-1(1+G)^{2(\alpha-1)}}{2(1+G-T)^{2\alpha-1}}-(\alpha-1)\ln(\frac{1+G}{1+G-T})\frac{(1+G)^{\alpha-1}}{(1+G-T)^{\alpha}}+\frac{1}{(1+G-T)}\right\}$$

$\frac{dF_3(T)}{dT}$ with respect to $T\in(0,T_w)$, yields:

$$\frac{dF_3(T)}{dT}=hDT\left[\frac{(1+G)^2}{2(1+G-T)^2}+\frac{1}{2}\right]+\frac{CD(1+G)}{(1+G-T)^2}+I_e PD+I_k CD(1+G)^2 T\left\{\frac{1}{2(1+G-T)^2}\left[1-(\frac{1+G}{1+G-T})^{2\alpha-1}\frac{(\alpha-1)(2\alpha-1)}{1+G}\right]\right\} \quad \text{(B2)}$$

Since $\alpha<1$ we have:

$$1-(\frac{1+G}{1+G-T})^{2\alpha-1}\frac{(\alpha-1)(2\alpha-1)}{1+G}>1-\frac{(\alpha-1)(2\alpha-1)}{1+G-T} \quad \text{(B3)}$$

As mentioned in assumptions, $\theta(t)=\frac{1}{1+G-t}$ for $t\in[0,T]$ and since $\theta(t)<1$ we have $\theta(T)=\frac{1}{1+G-T}<1$, then $\frac{(\alpha-1)(2\alpha-1)}{1+G-T}<1$ and consequently $1-\frac{(\alpha-1)(2\alpha-1)}{1+G-T}>0$ which gives $\frac{dF_3(T)}{dT}>0$. Therefore, $F_3(T)$ is a strictly increasing function of T in the interval $(0,T_w)$. From Equation (B.1) we know that $\lim_{T\to T_w^-}F_3(T)=\Delta_3$ and $\lim_{T\to\infty}F_1(T)<0$. Thus if $\lim_{T\to T_w^-}F_3(T)=\Delta_3>0$, by using the Intermediate Value Theorem, there is a unique $T_3\in(0,T_w)$ which gives $F_3(T)=0$. Moreover, taking the second derivative of $TRC_3(T)$ with respect to T at the point T_3 yields:

$$\frac{d^2TRC_3(T)}{dT^2}\Big|_{T_3}=\frac{D}{T_3}\left\{\frac{(h+C)(1+G)^2}{2(1+G-T_3)}+\frac{h}{2}+I_e P+I_k C(1+G)^2\left\{\frac{1}{2(1+G-T_3)^2}\left[1-(\frac{1+G}{1+G-T_3})^{2\alpha-1}\frac{(\alpha-1)(2\alpha-1)}{1+G}\right]\right\}\right\}>0 \quad \text{(B4)}$$

Therefore, $T_3\in(0,T_w)$ is the unique optimal solution for $TRC_3(T)$.

However, if $\lim_{T\to T_w^-}F_3(T)=\Delta_3<0$, then for all $T\in(0,T_w)$, we have $F_3(T)<0$. As a result for all $T\in(0,T_w)$ also $\frac{dTRC_3(T)}{dT}=\frac{F_3(T)}{T^2}<0$. Then we can claim that $TRC_3(T)$ is a strictly decreasing function of T in the interval $(0,T_w)$. Subsequently, there is no T in the open interval $(0,T_w)$ under which $TRC_3(T)$ is minimized and this completes the proof.