Using Data Envelopment Analysis in Markovian Decision Making

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Abstract

This paper introduces a modelling framework which combines Data Envelopment Analysis and Markov Chains into an integrated decision aid. Markov Chains are typically used in contexts where a system (e.g. staff profile in a large organisation) is at the start of the planning horizon in a given state, and the aim is to transform the system to a new state by the end of the horizon. The planning horizon can involve several steps and the system transits to a new state after each step. The transition probabilities from one step to the next are influenced by both organisational and external (non-organisational) factors. We develop our generic methodology using as a vehicle the homogeneous Markov manpower planning system. The paper recognizes a gap in existing Markovian manpower planning methods to handle stochasticity and optimization in a more tractable manner and puts forward an approach to harness the power of DEA to fill this gap. In this context, the Decision Maker (DM) can specify potential anticipated future outcomes (e.g. personnel flows) and then use DEA to identify additional feasible courses of action through convexity. These feasible strategies can be evaluated according to the DM's judgement over potential future states of nature and then employed to guide the organisation in making interventions that would affect transition probabilities to improve the probability of attaining the ultimate state desired for the system. The paper includes a numerical illustration of the suggested approach, including data from a manpower planning model previously addressed using classical Markov modelling.

Keywords: Data Envelopment Analysis, Markov Manpower Planning, Markov processes, Efficiency, Goal Programming

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1. Introduction and literature review

This paper integrates Markov Chains (MCs) and Data Envelopment Analysis (DEA) into a synergistic decision aid methodology. The methodology developed exploits the aspects of DEA to generate alternative courses of action and to identify the best among them, in order to simplify an otherwise stochastic framework and enhance the usefulness from the information that a MCs approach can yield as a decision aid. As a vehicle for integrating the use of DEA into a MCs based approach we use the Markovian human resource planning context. The use of a specific decision context facilitates the communication of the approach without detracting from its generalisability as the approach can be readily adapted to other decision contexts as will be discussed later.

1.1 Data Envelopment Analysis

Data Envelopment Analysis (DEA), is a non-parametric mathematical programming approach, that has been used widely to evaluate the relative performance (relative efficiency) of homogeneous entities, that is, operating units that use similar resources (inputs) to produce comparable products or services (outputs). One key feature of DEA is that under minimal assumptions it can use the data from observed operating units to generate other feasible in principle operating units even if not observed in practice. Another key feature of DEA is that it is a *boundary method* in that out of the units observed or created it can identify the most 'efficient' ones. That is, it estimates at operating unit level, maximum output levels for given resources or minimum input levels for given output levels. It is these two properties of DEA that we shall deploy in the context of a MCs methodology to identify interventions that would deliver outcomes best compatible with the Decision Maker's aims. Since the publication of the seminal paper by Charnes, Cooper and Rhodes (1978) who took forward ideas from a paper by Farell (1957) on the measurement of efficiency, the literature on DEA has flourished. The original idea was to provide a methodology that would yield a measure of efficiency of a Decision Making Unit (DMU) relative to a set of comparable DMUs in not for profit organisations. However, since then, DEA has evolved into a modelling approach to efficiency and productivity analysis which yields much additional information for exploring the production space and managing performance. For instance, it identifies benchmark DMUs (peers) inefficient units may draw best practice from to improve performance, it identifies targets inefficient units may seek to attain, the nature of returns to scale prevailing at a point of the efficient frontier, productivity change over time and much more. Due to its strength, applicability and minimal a priori assumptions, DEA has seen rapid growth and widespread acceptance in the last forty years, both in theory and practice. For more details on DEA see Thanassoulis (2001), Cooper (2005), Cooper et al. (2007a), Thanassoulis et al. (2008), Liu et al. (2013), Kao (2014) and Portela and Thanassoulis (2014). A more recent survey on the practice and progress of the theory and applications of DEA can be found in Emrouznejad and Yang (2018).

1.2 The Markovian manpower models and their use

The variability inherent in all processes involving human interaction influences the ability to forecast and eradicate the risks within an organization. As mentioned in a recent study on a Markov model for manpower prediction (Hrustek et al. 2020), in order to reduce business failures, various mathematical and probabilistic instruments have been developed with the purpose of mitigating these risks. MC models are among those methods, with a broad range of applications including the investigation of the mobility of personnel and the management of human potential within organizations. Hrustek et al. (2020) analyse a case study of an ICT service and use an irreducible, ergodic MC to predict numbers of people in the main staff categories thus assisting the management to plan ahead for budget requirements. As pointed out in a paper about Markovian manpower planning in the armed forces, Škulj et al. (2008), "the skills needed to perform assigned tasks are acquired through experience and training, shortfall or surpluses of skilled staff might be costly and inefficient". To cope with stochastic fluctuations of future needs, predictions and strategies need to be in place. Often estimates are based on previous experiences. Nevertheless, often experience alone does not suffice without the application of appropriate quantitative models and the associated analyses.

MCs have been used in many cases in the last fifty or so years in stochastic decision contexts. These include health care, corporate manpower planning, defence, educational establishments, civil services and, more recently, even in machine learning and artificial intelligence (AI). As McClean (1991) and Smith and Bartholomew (1988) mention, manpower planning can be traced back to 1779 when John Rowe used an actuarial model to plan careers in the Royal Marines. Pioneering in the area are considered the students' enrolment model of Gani (1963), the work of Young and Almond (1965) on predicting distributions of staff, the volumes edited by A.R Smith (1971, 1976 and 1980) on manpower planning systems, manpower planning in the civil services and corporate manpower planning, the classic book by S. Vajida (1978) on the mathematics of manpower planning and the seminal book by Bartholomew (1982). It is also not a coincidence at all that, in a series of papers, the pioneers of goal programming and DEA, A. Charnes and W.W. Cooper with their colleagues (1968, 1972, 1973, 1976) set the fundamentals for *predictive* and *normative* manpower modelling for civil services and/or corporate settings, using both probabilistic approaches and mathematical programming techniques, including multiple objective optimization. They even considered crucial issues that are still at the top of the the social agenda of our times, such as "ensuring equal employment opportunity representation of social groups within the organization that matches their representation in the environment surrounding the organization". In addition, Hopkins (1980) used a Markov chain based model as an aid in setting realistic numerical goals for the employment of women and minority persons in a real university environment.

The key features of Markov chains as an instrument for investigating future distributions of populations across states is that the recruitment, the internal transitions and the attrition probabilities drive the whole process. Manpower in organisations of this type, consists of sub-groups who interact purposefully and are usually stratified into network structures. Typically, state vectors (also called stock vectors) are used to describe the distribution of people (numbers of employees) that reside in various states of the system (groups, grades etc.) based on miscellaneous attributes such as job positions, departments, length of stay or age, skills, job description etc. These sub-groups are sometimes hierarchical in the sense that the grades correspond to promotional opportunities. The state vectors change in time according to flows which are typically regulated by internal transfers, wastage due to attrition, and recruitment of newcomers. The flow probabilities reflect, in most decision contexts, stochastic outcomes of purposeful actions such as recruitment and retention policies in manpower planning and medical treatments and their impact on health outcomes of a population.

In a MCs approach to human resource planning staff levels always converge in some asymptotic form when internal probabilities and recruitment vectors are kept constant over time. However, in reality, organizations exercise control over their personnel strategies to guide the process towards desired staff levels. In this respect, one of the many problems that has been given attention to in the Markov manpower literature is the problem of attaining desired staff levels, and in particular the achievement of desired structural configurations of personnel, using control variables such as recruitment flows, internal transfers, retirement and at times redundancies. The Markovian modelling framework lends itself naturally to this kind of personnel planning since it encompasses staff flows reflected in transition probabilities. The target is to use Markovian models in various forms to describe personnel mobility (forecasting aspect) and to support the preconditions to create feasible, satisfying or even optimal policies for attaining or maintaining appropriate population structures (normative aspect).

Formally, MC based manpower models are divided into two main streams; ***explorative and normative.*** *The* ***explorative*** (*descriptive, predictive*) models are used to get insight into the way a system operates (descriptive) and how it might respond to various interventions (predictive). ***Normative*** *(optimization)* models are used to identify optimal decisions based on control variables such as recruitment, training, promotional rates or retention. Markovian human resource models fall in the exploratory category and are used for predicting the population levels of organizations, contingent on alternative scenarios of the future (Wang, 2005). The solution of such stochastic models delivers insights, predictions and foresights regarding the organization's personnel and can guide purposeful interventions to achieve various goals towards planning and controlling a system's structure. A good account of baseline modelling tools can be found in two influential texts: Bartholomew (1982) and Bartholomew et al. (1991). Other approaches of stochastic models in manpower planning using MCs can be found in Vassiliou (1982) on the nonhomogeneous Markov system, Vassiliou and Tsantas (1984) on one step maintainability, Vassiliou and Georgiou (1990) on asymptotic behaviour, McClean and Montgomery (1999) on semi-Markov models, Papadopoulou and Vassiliou (1994, 1999) on semi-Markov asymptotic theory, Nilakantan and Raghavendra (2005) on proportionality constraints of attainability, Ossai and Uche (2009) on departmentalized manpower structures, Guerry (2011) on the hidden heterogeneity in manpower systems, and more recently, Dimitriou and Georgiou (2020) who elaborated on departmental mobility in continuous time multivariate Markov settings. From the vast body of the relevant literature we distinguish two papers that are somewhat closer to our proposed approach, namely De Feyter and Guerry (2009) and Nilakantan (2015). Nilakantan (2015) established an effectiveness measure for the evaluation of staffing policies in Markov manpower systems, in relation to the career growth prospects afforded by the system to its members. Our paper has a slight similarity in that it evaluates policies specified up front. However, we measure the effectiveness of proposed manpower flows rather than the career prospects at employee level using management aspirations. In De Feyter and Guerry (2009) the authors propose an approach for the evaluation of recruitment strategies. We too address the issue of identifying efficacious recruitment strategies but unlike De Feyter and Guerry (2009) who use probabilities and fuzzy membership functions, we use DM judgements and preferences which are more user-friendly concepts. We compare later their approach to ours using an example from their paper.

A number of applications and case studies have emerged in the literature in tandem with the theoretical development of the field as outlined above. A frequent area of application of MCs is that of **academic manpower**

planning, see Ledwith (2019) for a recent review. In earlier studies, such as Bleau (1981) and Hackett et al. (1999) we see that Markov approaches provide additional insight when planning for personnel in short or midterm horizons. In the academic setting the states of the Markov chain usually represent teaching and research staff by academic discipline and grade. The transitions in, out, and within the states would reflect stochastic outcomes of a college's policies on recruitment, attrition and promotion. The stochasticity is of course also influenced by random events and factors not in the gift of the University, such as the external market for academic staff, Government policies on education etc.

Another major area of application of manpower planning is that of **military forces**. There exist several applications of both *forecasting* and *normative* MC methodologies for projecting career progressions and for modelling personnel's aggregate mobility behaviour. For a good review see Wang (2005) and Ledwith (2019). Among the first to employ Markov chains to determine the structure of armed forces was Brothers (1974). In 1984, the RAND Corporation issued a report regarding the use of Markov chain models to predict the probability of officers staying in the Air Force (Gotz and McCall, 1984). Another well-known framework was put forward by Gass et al. (1988) and Gass (1991), who proposed Markov chain models and goal programming techniques to forecast flows and determine optimal policies using as personnel states combinations of ranks, skills, operational units and length of service. Hall (2009), built a Markov model to determine optimal policies regarding officers' decision for retirement (see also Cashbaugh et al. 2007). Škulj et al. (2008) presented a case study using Markovian manpower planning for the Slovenian armed forces to produce projections for several years ahead and to apply appropriate policies for achieving specific staff level goals. Van Utterbeeck et al. (2009), presented a combined simulation and Markov chain model for human resource management in the Belgian Defence Forces. Recently, Zais and Zhang (2016), developed a Markov chain model to predict individualized stay/leave decisions within the US Army and Ledwith (2019) presented a Markov model with absorbing states to forecast educational composition according to arrival and internal transition probabilities for a military academy.

Another field of application of MC manpower planning is that of **healthcare organizations**. Early applications of Markovian models in *health care* can be found in Shuman et al. (1971) and Smith et al. (1976) who developed mathematical programming models for investigating optimal staffing in health services. Trivedi et al. (1987) focused their attention on Markov models for health care manpower supply predictions and McClean et al. (2011) developed a modelling framework that combines Markov chains with simulation to investigate the whole care system of stroke patients for Belfast City Hospital. Josiah (2014) introduced a Markov forecasting model as a tool for Health Corps Administrators to forecast inventory levels across ranks and subspecialties. Also, Lagarde and Cairns (2012) used a Markov model to examine the dynamics of movements of health care workers in the professional labour market in South Africa. In the same direction, Srikanth (2015) modelled the progression of diabetic retinopathy estimating the time a patient spends on each one of five stages of the disease from mild retinopathy through to moderate, severe, PDR (Proliferative Diabetic Retinopathy) and ultimately blindness (single or double). Using new patient arrivals to each stage and transitions of existing patients consequent on any medical interventions, the number of patients at each stage of the disease at future points in time can be estimated. This information clearly has important implications both for prognosis at patient level but also for resource planning at health service provider level. In a recent review of the relevant literature, Bartelt‐Hofer et al. (2020), report that "*Markov models using transition states were the most common type of modelling approach. Cost-utility models using a mid- to long-term time horizon with a national payer perspective were the most frequent type of economic evaluation identified*". A good account of patient flow modelling of healthcare delivery and performance analysis using various techniques such as queueing models, Markov chains, simulation modelling and statistical methods, can be found in Bhattacharjee and Ray (2014).

1.3 The research gap and the contribution in this paper

Although as noted above there is strong evidence that the HR community recognizes the need for strategic manpower planning, it seems that barriers such as preoccupation with short-term activities and the complexity of uncertainty hinder the wider adoption of MC manpower planning models. Taylor (2005) points out that when business decision makers try to predict a company's internal labour supply they lean towards empirical judgements and intuition and not on a thorough statistical analysis. Škulj et al. (2014) mention that other reasons raised by researchers in the field of management are hostility to statistical (quantitative) techniques, preference of intuitive judgment, ignorance, and short-term mentality. The reasons put forward in Taylor (2005) and Škulj et al. (2014) are clearly in line with the major findings of the report by Johnson and Brown (2004) attributed to the rapid change of the environment and high level of uncertainty in contemporary business environment. As Oczki (2014) emphasises, the doubts concerning the application of more formal (quantitative) methods in systematic labour force prediction and optimization, should not hinder the fact that these approaches, in the long run, can significantly contribute to an organization's success.

In this respect, the method developed here aims to make the handling of uncertainty in an MC context more tractable. The majority of MCs approaches in manpower planning focus on examining the population's (mostly) stochastic flowing mechanism and attempt to determine an optimal strategy (e.g. recruitment or promotion), that attains or maintains specific targets in terms of personnel. However, to the extent that these mobility patterns can be influenced by the actions of an agent (University in the case of employees, clinicians in the case of patients, the Department of Defence in the case of military personnel etc.) the issue arises how to practically identify alternative actions an agent can take to achieve desirable outcomes which are stochastic in nature. The approach needs to be tractable so as to be more accessible to non-specialist managers and practitioners. This paper recognizes the gap in existing Markovian manpower planning methods to handle stochasticity and optimization in a more tractable manner and puts forward an approach to harness the power of DEA to fill this gap. Within DEA it is possible to specify a set of potential feasible recruitment strategies and create a space of an infinite number of alternative strategies as convex combinations of those specified. These feasible strategies can be evaluated to identify those best suited to the personnel targets. This information can then be used to design interventions that would influence the flows of the Markov Chain concerned to lead to the desired outcomes. Our approach, in effect does not require precise probabilities and uses instead judgement of likelihood of state of nature. These likelihoods are converted to weights in the DEA model making the solution much more amenable in practical situations than analytical solutions in a purely stochastic context.

The rest of the paper is organized as follows. The next section, Section 2, presents background theory and develops a radial version of the hybrid DEA - Markovian manpower model. Section 3 puts forward an alternative, additive version, of the hybrid DEA-Markov model, offering the user the ability to incorporate additional preferences and likelihood information over potential manpower flows. Section 4 presents an ex post measure of the efficacy of alternative personnel flows. Section 5 illustrates the methodology developed using numerical data. The paper concludes in section 6.

2. Theoretical framework

2.1 The Markov manpower system and the classic problem of attainability

Markov manpower models stretch more than four decades back. The cornerstone of human resource planning at the macro level is the classification of the workforce, into subgroups according to some common characteristics. These subgroups are called classes, states, or grades of the Markov Chain and their levels, (stock levels), characterise the population structure of the organisation. The mobility of staff is investigated by considering the various forms that stock vectors might take as a result of rules, functional assumptions, interventions and randomness acting on the system. The majority of these models use discrete time scales and a common notation for the number of states (grades) of a system is k. The personnel stocks (the state vector) at any point in time t, is denoted by a row vector ***N***(t), for t = 0, 1, 2… The initial population structure N(0) typically comprises non negative real numbers and it is a designated known initial vector or a vector in a presumed steady state.

In general, for any specific starting point at time t-1, this initial stock vector is denoted by ***N***(t-1). Then, the baseline non-homogeneous Markov system is realized by the difference equation in (1).

$$\boldsymbol{N}(t) = \boldsymbol{N}(t-1)\boldsymbol{P}(t-1) + [\boldsymbol{N}(t-1)\boldsymbol{p}'_{k+1}(t-1) + \Delta T(t-1)]\boldsymbol{p}_0(t-1) \qquad (1)$$

where, $\boldsymbol{N}(t-1) = \left[N_1^{t-1}, N_2^{t-1}, \dots, N_k^{t-1}\right]$ is the known stock vector of the organization's classes at time t-1 and $\boldsymbol{N}(t) = [N_1^t, N_2^t, \dots, N_k^t]$ denotes the one step resulting vector of the expected population levels, based on a discrete time scale t=0,1,2,… The transition matrix denoted by ***P(t-1)*** contains the probabilities of internal mobility (i.e. flows amongst the k grades of the system) and obey the Markov property (no memory). For simplicity, we assume that the transition matrix is homogeneous in time, thus ***P(t)*** can be substituted by ***P*** for all t=0,1,… The transition matrix is *substochastic* since there is an additional state, denoted by k+1, which accepts attrition of all types (i.e. retirement, dismissal, voluntary leaving etc). In this respect, $\boldsymbol{p}'_{k+1}$ is a column vector including the probabilities of attrition (calculated by 1-$\sum_{j=1}^{k} p_{ij}$ for every i=1,2,…k) where p_{ij} is the transition probability from state i to state j. $\boldsymbol{N}(t-1)\boldsymbol{p}'_{k+1}$ is the expected number of leavers (attrition) during the time interval [t-1, t). ΔT(t-1) denotes the probable expansion of the system (theoretically, ΔT(t-1) cannot be negative.) We then have a probability recruitment vector, which often serves as the control variable if our aim is to drive the system towards some desirable population structure in a single or in several steps in time (or even asymptotically). This row recruitment vector, denoted by $\boldsymbol{p}_0(t\text{-}1)$, is essentially a stochastic vector containing the probabilities for both types of recruitment, that is for newcomers due to expansion $\Delta T(t-1)$ and attrition replacements due to $\boldsymbol{N}(t-1)\boldsymbol{p}'_{k+1}(t-1)$, who all enter the system in some of its k grades. Using a homogenous in time transition matrix, we get the well-known simpler form of the homogeneous model, depicted in (2).

$$\boldsymbol{N}(t) = \boldsymbol{N}(t-1)\boldsymbol{P} + [\boldsymbol{N}(t-1)\boldsymbol{p}'_{k+1} + \Delta T(t-1)]\boldsymbol{p}_0(t-1) \qquad (2).$$

As it is obvious, the resulting expected structure ***N(t)***[2] depends on the population structure at time (t-1), on the internal flows regulated by the transition matrix ***P*** and by a recruitment distribution stochastic vector denoted by $\boldsymbol{p}_0^{(t-1)}$.

In the context of the foregoing discussion we take the stock vector as reflecting job positions or placeholders in general rather than specific individuals. It is true that in the case of specific persons everybody leaves in the long run, ending up in the absorbing state k+1. That is not the case for the stock vector of placeholders (Vassiliou et al. 1990, Vassiliou and Papadopoulou, 1992). The origin of this concept lies in Bartholomew (1982) where it is stressed that "an open system in which gains and losses were equal could be treated as closed. Each person who leaves can be paired with a new entrant and the two changes treated as one. Thus a transition from grade i to j can either take place within the system or by loss from grade i and replacement to grade j ...". Since we observe the system at a discrete time scale, it is common to assume that the interval between t-1 to t provides enough time to accommodate and absorb any lag between attrition and recruitment.

We develop our approach assuming that the organisation aims at a desired population structure denoted ***N**** to be attained by a given time horizon. The decision maker wishes to identify interventions that will influence the transition patterns so as to attain ***N**** in one step. Clearly the use of one step has implications for the duration of the planning horizon, in that the longer the planning horizon the more the steps needed to attain ***N****. Thus, the specification of ***N**** can be such that the planning horizon is of a corresponding duration. Assuming then one step planning horizon the model of equation (2) takes the following form for a specific recruitment vector $\boldsymbol{p}_0(t-1)$:

$$\boldsymbol{N}^* = \boldsymbol{N}(t-1)\boldsymbol{P} + [\boldsymbol{N}(t-1)\boldsymbol{p}'_{k+1} + \Delta T(t-1)]\boldsymbol{p}_0(t-1) \qquad (3).$$

We can simplify the formulation in (3) by using recruitment flows instead of probability recruitment vectors. This is shown in equation (4) where $\boldsymbol{N}(t-1)\boldsymbol{P}$ is as in (3) and $\boldsymbol{R}(t-1)$ denotes a *recruitment flow vector,*

$$\boldsymbol{N}^* = \boldsymbol{N}(t-1)\boldsymbol{P} + \boldsymbol{R}(t-1) \qquad (4).$$

This formulation provides a much more flexible pattern of flow than those in equation (3). Conceptually ***R(t-1)*** is equal to $[\boldsymbol{N}(t-1)\boldsymbol{p}'_{k+1} + \Delta T(t-1)]\boldsymbol{p}_0(t-1)$ and in that sense is subject to uncertainty. However, in our approach we shall use alternative flow vectors (candidate policies) to cover the domain of potential flow vectors drawing on the knowledge of the user about the decision context at hand. When ***R(t-1)*** equals $[\boldsymbol{N}(t-1)\boldsymbol{p}'_{k+1} + \Delta T(t-1)]\boldsymbol{p}_0(t-1)$ it means that we really do need to secure the covering of vacancies that arise during a period and we could also have some expansion designated by *ΔT*(0). It is interesting to rewrite (4) in the equivalent form depicted in (5) where **e'** is a column vector of ones.

$$\boldsymbol{N}^* = \boldsymbol{N}(t-1)\boldsymbol{P} + \boldsymbol{R}(t-1)\boldsymbol{e}'\frac{\boldsymbol{R}(t-1)}{\boldsymbol{R}(t-1)\boldsymbol{e}'} \qquad (5).$$

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[2] Another form of equation (1) is the following: $\boldsymbol{N}(t) = \boldsymbol{N}(t-1)\boldsymbol{Q}(t-1) + [\Delta T(t-1)]\boldsymbol{p}_0(t-1)$

where $\boldsymbol{Q}(t-1) = \boldsymbol{P}(t-1) + \boldsymbol{p}'_{k+1}(t-1)\boldsymbol{p}_0(t-1)$ is in essence a stochastic matrix that includes the probabilities that regulate both internal transitions and recruitment attributed to attrition. This form has been widely used in the literature to study various issues related to system evolving either on discrete time points or in its asymptotic forms and its variability properties, depending on the properties of the matrix ***Q*** (which can be for example, regular, periodic or cyclic depending on the embedded markov chain).

In this case, $\frac{\boldsymbol{R}(t-1)}{\boldsymbol{R}(t-1)\boldsymbol{e}'}$ is in fact the corresponding recruitment mix equivalent to $\boldsymbol{p}_0(t-1)$ in (3) and $\boldsymbol{R}(t-1)\boldsymbol{e}'$ represents the total population recruited.

Clearly we cannot be certain $\boldsymbol{N}^*$ can be attained in one step for a specific recruitment policy $\boldsymbol{p}_0(t-1)$ or recruitment flow $\boldsymbol{R}(t-1)$, and so we cannot be sure equation (3) or (4) is feasible. See for example (Bartholomew 1982, Vassiliou and Tsantas, 1984) for the necessary and sufficient conditions in order to have a feasible stochastic recruitment vector $\boldsymbol{p}_0(t-1)$ that drives the system exactly to $\boldsymbol{N}^*$.

In order to allow for the possibility that the equation in (4) may not be feasible for a given recruitment flow $\boldsymbol{R}(t-1)$ we allow for the under or over attainment of components of the target $\boldsymbol{N}^*$ by using deviation variables as in the goal programming context. This is shown in (6),

$$\boldsymbol{N}^* = \boldsymbol{N}(t-1)\boldsymbol{P} + \boldsymbol{R}(t-1) + \boldsymbol{d}^- - \boldsymbol{d}^+ \quad (6).$$

The variables $\boldsymbol{d}^-$ and $\boldsymbol{d}^+$ are vectors of negative or positive deviations from the target $\boldsymbol{N}^*$. Equation (6) is always feasible since any excess or shortfall from the levels in $\boldsymbol{N}^*$ are captured in the deviational variables $\boldsymbol{d}^-$ and $\boldsymbol{d}^+$**.** This approach linearizes the model and also caters not only for attrition and/or expansion, but also for probable contraction of the system. Since the aim is to attain the desired structure $\boldsymbol{N}^*$, typically both types of deviational variables are undesirable. As we will see later though, weights are used in our model to reflect the DM preferences. The model to be developed will have a minimisation objective function which will ensure that at its optimal solution at most one of the deviational variables in each constraint of (6) can be positive.

The ultimate aim is to attain a desired stock vector $\boldsymbol{N}^*$, from an initial structure $\boldsymbol{N}(0)$ (setting without loss of generality t-1 = 0, the initial time point). We address this aim by assuming that the management of the organization can come up with a set of potential recruitment flows to be explored for their compatibility with $\boldsymbol{N}^*$ within the time horizon being considered. The flows management will specify, will reflect the range of possible combinations of the stable internal transitions and departures along with recruitments being planned. These managerial views will draw on historical data on transitions, staff attrition and recruitments as a basis for projecting alternative visions of recruitment flows into the future. We will take advantage of the DEA ability to then create an 'infinite' set of virtual recruitment flows to be explored for attaining $\boldsymbol{N}^*$ from $\boldsymbol{N}(0)$. More precisely, we will use DEA to pursue simultaneously two aims: On the one hand the *potential recruitment vectors will each be evaluated for their own merit to guide the system to the desired personnel state* denoted $\boldsymbol{N}^*$; on the other DEA will enable us to *identify other potential recruitment flows, not initially specified by management, which may be even more efficacious* in leading to $\boldsymbol{N}^*$ or its proximity.

2.2 The development of a hybrid Markov-DEA model

In the general form of DEA we have a collection of *n* Decision Making Units (DMUs) which use a set on *m* inputs (e.g. personnel, materials) to secure *s* outputs (e.g. students taught, research outputs delivered etc.). The aim is to assess the relative efficiency of each DMU, in terms of minimum input levels that could support its output levels, compared to other DMUs in the set. (Output efficiency is defined in an analogous manner.) Full details of DEA can be found in a number of texts, including Thanassoulis (2001) and Cooper et al. (2007a). We shall deploy the framework of DEA within the context of a Markov Chains approach to manpower planning.

Let us denote by x_{ij} and y_{rj} respectively the level for the i-th input, *i*=1,…,*k* and the r-th output, *r*=1,…,*m* for DMU j. The technical input efficiency of a specific DMU denoted by j_0 is the optimal value of θ in model (7). Any feasible set of the λ values in (7.1) and (7.2) identifies through a convex combination a virtual or real DMU within the feasible production possibility set created using data from the real DMUs *j*=1,2,….,*n*. Equations (7.1) assure that the virtual DMU uses no more than a fraction θ of the inputs of j_0 and equations (7.2) ensure that the virtual DMU secures at least the output levels of DMU j_0. This model assumes variable returns to scale in that DMU j_0 can be assessed only relative to other DMUs that have the same scale size as itself. The optimal value θ* in the objective function will reflect the lowest fraction to which the inputs of DMU j_0 can be reduced without detriment to its output levels, using benchmarks from the n DMUs in the set. In this sense θ* is the (input) efficiency of DMU j_0.

$$\text{Min } \theta - \varepsilon\left[\sum_{i=1}^{k} s_i^- + \sum_{r=1}^{m} s_r^+\right] \tag{7}$$

subject to:

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{ij_0} \text{ for } i = 1,2,\ldots,k \tag{7.1}$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rj_0} \text{ for } r = 1,2,\ldots m \tag{7.2}$$

$$\sum_{j=1}^{n} \lambda_j = 1 \tag{7.3}$$

$\lambda_j \geq 0, j = 1,2,\ldots n, s_i^-, s_i^+ \geq 0$ for all i and r, θ unrestricted and ε is a non-Archimedean infinitesimal.

Using ε<<1 in the objective function, gives pre-emptive priority to the minimisation of θ and thus in effect the model is solved in two phases. In the first phase, priority is given to the minimization of the 'radial efficiency' measure θ. Consequently the model identifies a point that uses the lowest proportion θ^* of the input levels of DMU j_0 when output levels are at least as high as those of DMU j_0. Then maintaining the optimal value θ* of θ the model seeks the maximum sum of the slack values(s_i^- and s_r^+) to identify feasible improvements to input and output levels after input levels are reduced proportionally to the fraction θ*. (This solution process is often carried out in two-stages, avoiding the need to use ε, a non-Archimedean infinitesimal; e.g. see the manual for the PIM DEA software www.deasoftware.co.uk). If θ*=1 and no positive optimal slacks are found DMU j_0 is identified as 'Pareto' efficient. However, even when θ*=1, if at least one positive optimal slack exists, the DMU is not Pareto efficient though it is lying on the boundary of feasible production points. DMU j_0 is clearly not efficient if θ*<1, and even more so if additionally any optimal slack is positive. Note that model (7) has always a feasible solution in which θ=1, and thus at its optimal solution we will have $\theta^* \leq 1$.

We return now to the development of a hybrid DEA model that includes the homogeneous Markov system and can be used to establish a framework for assessing a set of potential alternative recruitment flows to attain a target manpower structure denoted ***N****. In this respect, we need to define our DMUs, their inputs and their corresponding outputs. We must also select input or output orientation accordingly and incorporate the DEA framework within the Markov population model. First, we define the DMUs as the set of (n) potential recruitment flows specified by management. Each alternative recruitment flow $\boldsymbol{R}_j$, $j = 1,2, \dots, n$ is a k-dimensional vector of staff categories, containing non negative entries which we will treat as 'inputs' in the DEA framework. The recruitment flows $\boldsymbol{R}_j$, $j = 1,2, \dots, n$ are user specified. They are the instrument by which the Decision Makers (DMs) can reflect the range of alternative person flows that can result from internal transitions and attritions and external in-flows. The DM will construct these flows drawing on historical transition and attrition probabilities, combined with judgement of the evolving internal human resources and external recruitment environment going forward. It is important to note that the DM is not expected to predict recruitment flows. Rather, using judgement, historical data and evolving strategies to lay out as wide as possible a range of potential recruitment flows $\boldsymbol{R}_j$, $j = 1,2, \dots, n$ to ensure that some average of the strategies specified will materialise. It is DEA that will assess the efficiency of alternative averages of the specified recruitment flows $\boldsymbol{R}_j$, $j = 1,2, \dots, n$ to lead to the target ***N*.*** The recruitment flows $\boldsymbol{R}_j$ in keeping with DEA can only take non negative values. However, any reductions in personnel desired within ***N**** can still be catered for in our model through its goal programming structure which allows for both under and over attainment of components of ***N*.***

The proposed DEA model will have a notional output level of 1 in the spirit of the Benefit of the Doubt model (e.g. Karagiannis and Karagiannis, 2018). We shall explain the notional output level below after we develop the formulation of the model further. We use an input orientation, consistent with the notion that as staff costs money, ceteris paribus, the lower the staff flow levels the better.

Let us denote the alternative recruitment flows $\boldsymbol{R}_j$, $j = 1,2, \dots, n$ where $\boldsymbol{R}_1 = [R_{11}, R_{21}, \dots, R_{k1}]$, $\boldsymbol{R}_2 = [R_{12}, R_{22}, \dots, R_{k2}]$...$\boldsymbol{R}_n = [R_{1n}, R_{2n}, \dots, R_{kn}]$, and the target population structure $\boldsymbol{N}^* = [N_1^*, N_2^*, \dots, N_k^*]$. Thus R_{ij} is the ith input level of the jth proposed recruitment flow specified by the user and N_i* the target level for input (staff category) i. For convenience, we shall use the notation ***N***(0) for the initial vector of stock levels. We start with an initial population structure ***N***(0) and we want to evaluate the relative ability (**efficacy**) of the potential recruitment flows $\boldsymbol{R}_j$, j=1…n in respect of attaining the ideal population structure ***N****. We can use for this purpose the generic model in (7). The input constraints (7.1), when $\boldsymbol{R}_{j0}$ is the object of the assessment takes the form:

$$\sum_{j=1}^{n} \lambda_j R_{ij} + s_i^- = \theta R_{ij0} \qquad \text{for } i = 1,2, \dots k \qquad (8).$$

Here, the radial measure θ will capture the scope, if any, for reducing all the levels of staff in $\boldsymbol{R}_{j0}$, keeping their mix constant, in pursuit of ***N****. This scope can only be ascertained when we take into account the starting and the ideal terminal staff levels. This is done by recourse to the Markov manpower system detailed expression (6), which in vector form is reproduced below:

$$\boldsymbol{N}^* = \boldsymbol{N}(0)\boldsymbol{P} + \boldsymbol{R}_{j0} + \boldsymbol{d}_{j0}^- - \boldsymbol{d}_{j0}^+.$$

The i^{th} component of the vector above takes the form:

$N_i^* = \sum_{l=1}^{k} N(0)_l p_{il} + R_{ij0} + d_{ij0}^- - d_{ij0}^+$ for $i = 1,2,\dots,k$ (9).

We link expressions (8) and (9) by plugging into equation (9) the projection of $\boldsymbol{R}_{j0}$ on the efficient frontier, that is $\sum_{j=1}^{n} \lambda_j R_{ij}$ for $i = 1,2,\dots,k$ resulting in

$N_i^* = \sum_{l=1}^{k} N(0)_l p_{il} + \sum_{j=1}^{n} \lambda_j R_{ij} + d_{ij0}^- - d_{ij0}^+$ (10).

The formulation in (10) absorbs in the deviational variables any overshoots or shortfalls from ***N**** that could arise, contingent on the optimal projection of $\boldsymbol{R}_{j0}$ to the target in $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ (superscripts * indicate optimal values of λ). It is recalled that each flow $\boldsymbol{R}_j$ reflects a different variant perceived by the DMs of the historical internal and recruitment transition probabilities. This enables the DMs to map out the multiple potential future "states of nature" so far as staff level transitions, attritions and recruitments are concerned. Preserving the mix of personnel of each $\boldsymbol{R}_j$ when solving the DEA model signals that the DM has no granular level insights as to which components of $\boldsymbol{R}_j$ are more and which less likely to materialise as future states of nature. (We relax this assumption later.) The Markovian equations in (10) when integrated in a DEA model will yield the "best" virtual recruitment compatible with the user specified potential $\boldsymbol{R}_{j0}$. We refer to this as the best feasible projections $\boldsymbol{R}_{ij0}^*$ of $\boldsymbol{R}_{j0}$ where :

$R_{ij0}^* = \sum_{j=1}^{n} \lambda_j^* R_{ij} = \theta^* R_{ij0} - s_i^{*-}$ for i=1,2,…,k (11).

As already indicated, the DEA model we shall solve to derive $\boldsymbol{R}_{ij0}^*$ is input oriented with priority to preserve the flow mix in $\boldsymbol{R}_{j0}$. Thus our model will determine the optimal path to ***N**** when the historical transition probabilities combined with any personnel expansion or contraction aims are reflected in $\boldsymbol{R}_{j0}$. The optimal path is aimed to be, where feasible, at contracted levels of the mix of personnel reflected in $\boldsymbol{R}_{j0}$. To the extent that any slacks s_i^- take positive values at optimality they will reflect deviations needed from the strict proportionality captured in $\boldsymbol{R}_{j0}$ for reaching $\boldsymbol{N}^*$. Finally to create feasible in principle convex combinations of the personnel flows proposed by management we impose the familiar in DEA convexity constraint in (12).

$\sum_{j=1}^{n} \lambda_j = 1$ (12).

Thus our model is the classical Variable Returns to Scale model albeit with a notional 'output level' of 1 across all units captured in the convexity constraint. DEA models where DMUs have a common input or output level are usually referred to as *Benefit of the Doubt* (BoD) models (Lovell and Pastor 1999, Rogge et al. 2017, Karagiannis and Karagiannis, 2018). Such models act as instruments to compare on desirability a set of alternatives, personnel flows in our case, where the attributes of the alternative being assessed are given the most favourable weights possible in measuring its desirability compared to the other alternatives in the set.

Finally in the context of the basic DEA model in (7) the objective function of the hybrid DEA –Markov model is

Min $\theta - \varepsilon_s\{\sum_{i=1}^{k} s_i^-\} + \varepsilon_d\{\sum_{i=1}^{k}(d_{ij0}^- + d_{ij0}^+)\}$ (13).

The ε_s and ε_d are user specified weights. In order to give pre-emptive priority to the minimisation of θ, ε_s and ε_d should be set several orders of magnitude lower than the coefficient of θ. Notice that since we wish to attain the specific target population levels of ***N****, both groups of deviational variables, d_{ij0}^- *and* d_{ij0}^+, are included in the objective function to be minimised. The user could set different orders of magnitude for ε_s and ε_d for example to signify that the minimization of the deviational variables d_{ij0}^{-} *and* d_{ij0}^{+} is to take priority over the maximization of the slacks s_i.

Combining together (13), (12), (10) and (8) we arrive at the following hybrid DEA-Markov model to assess the efficacy of the potential recruitment flow vector $\boldsymbol{R}_{jo}$ relative to the ideal manpower structure $\boldsymbol{N}^*$:

$$\text{Min } \theta - \varepsilon_s\{\textstyle\sum_{i=1}^{k} s_i^-\} + \varepsilon_d\{\textstyle\sum_{i=1}^{k}(d_{ij0}^- + d_{ij0}^+)\} \qquad (14)$$

subject to

$$\textstyle\sum_{j=1}^{n} \lambda_j R_{ij} + s_i^- = \theta R_{ij0} \qquad \text{for } i = 1,2 \dots, k \qquad (14.1)$$

$$N_i^* = \textstyle\sum_{l=1}^{k} N(0)_l p_{il} + \sum_{j=1}^{n} \lambda_j R_{ij} + d_{ij0}^- - d_{ij0}^+ \qquad \text{for } i = 1,2 \dots, k \qquad (14.2)$$

$$\textstyle\sum_{j=1}^{n} \lambda_j = 1 \qquad (14.3)$$

$\theta, \lambda_j, s_i^-, d_{ij0}^-, d_{ij0}^+ \geq 0$ for all i and j.

The model in (14) is essentially in the spirit of the basic DEA model (7). It gives priority to a radial contraction factor θ of a proposed mix of recruitment but it is embedded in the Markovian manpower model in a goal programming setting. The goal programming approach is necessary as none of the ex-ante proposed recruitment flows, deployed to give expression to the future uncertain personnel flows, is guaranteed to lead to the desired ideal population structure ***N**** in one step. The model reflects the notion of attainability of potential recruitment flows in the Markovian manpower context and measures the relative efficacy of each one in that context, through the capacity to produce a projected flow vector that will take the system as close as possible to the target ***N****.

If at the optimal solution to (14) θ*=1 and all slacks and deviational variables are zero then $\boldsymbol{R}_{j0}$ is **fully efficient** in the sense that it leads to ***N**** directly as specified ab initio in combination with anticipated internal transitions. If θ* is not 1 and/or any one of the slacks s or deviational variables d_{ij0}^- *or* d_{ij0}^+ is positive, it means $\boldsymbol{R}_{ij0}$ does not lead to the ideal population structure ***N**** in one step. When a flow $\boldsymbol{R}_{jo}$ does have a radial contraction factor θ*=1 but it has some slacks and/or deviational variables positive at the optimal solution to (14) then some but not all components of ***N**** are attainable from ***N***(0) in one step, should $\boldsymbol{R}_{jo}$ materialise in reality. Perhaps the more interesting case, is the one of a potential flow vector $\boldsymbol{R}_{jo}$ where (14) yields a contraction factor θ*<1 while some slacks s and/or deviational variables d are positive. In this case, no component of ***N**** is attainable in one step from ***N***(0), should $\boldsymbol{R}_{jo}$ materialise in reality. When $\boldsymbol{R}_{jo}$ is not fully efficient the solution of the model through the target $R_{ij0}^* = \sum_{j=1}^{n} \lambda_j^* R_{ij}$ will identify a potential flow which is more efficacious than $\boldsymbol{R}_{jo}$ itself in terms of attaining in one step the target ***N****. The flows are states of nature and not in the gift of the DM. However, the DM through the strategies, policies and timings thereof that they adopt can influence the manpower flows. Thus, where the DM has deemed $\boldsymbol{R}_{jo}$ as a potential state of nature and model (14) yields a more efficacious $\sum_{j=1}^{n} \lambda_j^* R_{ij}$, the DM can modify the $\boldsymbol{R}_{jo}$-related policies, timings and strategies to favour as far as possible the materialisation of the flow depicted in $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ rather than $\boldsymbol{R}_{jo}$. **It is in this context that the hybrid DEA-Markovian model developed here constitutes a useful aid to the decision making process for manpower planning.**

It should be noted that it is possible that none of the alternative potential flow vectors ***R*** turns out to be fully efficient. Nevertheless, the goal programming formulation of DEA, by construction always yields a feasible solution. The hybrid DEA-Markov model proposed here represents a formulation of the Markovian population model to accommodate the attainability problem. In this context it is using a set of alternative potential recruitment policies specified by the DM drawing on historical transitions data and managerial objectives through recruitment, hiring or attrition. The notion of using alternative 'flows' will differ by Markovian context. For example in a healthcare context the recruitment flows ***R*** would represent patient flows to the stages of some chronic condition such as diabetic retinopathy. The flows as in manpower planning will not be at the gift of the care provider but will be influenced by the nature and timing of clinical interventions such as screening, treatment, comorbidities etc. The model will then be used as above to aid the development of interventions consistent with the best (most efficient) patient flows identified through model (14).

3. A non-radial variant of the hybrid DEA-Markov model

The hybrid DEA-Markov model as presented so far assesses the relative effectiveness of alternative recruitment flows ***R***, in leading to an ideal target ***N**** assuming the DM has no prior information as to which components of a flow $\boldsymbol{R}_j$ are more likely to materialise, and no preferences such that getting closer to some components of $\boldsymbol{N}^*$ may be more desirable than attaining others. In this section we relax these two assumptions. In some real life cases it is possible that the organisation might have a perception on the probabilities of materialisation on certain inflows in a stochastic setting, or varying degrees of preference to achieve at their ideal level different categories of personnel contained within ***N****. For example, in a Business School of a University where restructuring may be taking place, it may be more desirable to achieve lead researchers at full Professor level than junior teaching fellows at this juncture in order to then grow academic staff round core lead research themes. Moreover, in specifying alternative potential flows and taking into account the much stronger competition between institutions for lead researchers than junior academic staff, the components within flows ***R*** relating to lead researchers may be less certain than those relating to junior staff. We propose an **additive** variant of the hybrid DEA-Markov manpower model developed in the previous section in order to better reflect this decision context.

The additive model will afford us the facility to weight any deviations at component level both between $\boldsymbol{R}_j$ and its projection $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ on the one hand, and between the attained and the ideal target $\boldsymbol{N}^*$ on the other. The classical additive DEA model was first proposed by Charnes et al. (1985). We adapt the objective function to include two parts: the minimization of the undesired weighted slacks and the undesired weighted deviational variables. The weights are user specified. Those relating to the slacks s express the **<u>degree of confidence</u>** the DM has about the outturn of each component of $\boldsymbol{R}_{j0}$ which is the object of the assessment. The weights relating to the deviational variables reflect the **<u>relative importance</u>** the user attaches to attaining the number of staff desired at each category or level *k*. The weights in both cases need to take into account the implicit weighting each staff category may already have by virtue of different scale sizes of staff categories, should that be the case – e.g. fewer managers compared to shop floor personnel in say a factory. The additive hybrid DEA-Markov model is in (15).

For each staff category/level i, the optimal values of s^- and s^+ in model (15) reflect respectively the shortfall or surplus from the optimal projection $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ of the flow $\boldsymbol{R}_{j0}$ being assessed. The optimal projection is with respect to attaining $\boldsymbol{N}^*$, contingent on the weights z on the deviational variables d in model (15) (explained below). The higher the probability the DM believes to be of attaining the level of staff category i compared to that of m, the higher should be the weights attaching on the slacks s_i^-/ s_i^+ compared to those placed on s_m^-/ s_m^+ in model (15). This would prioritise the minimisation of s_i^-/ s_i^+ compared to s_m^-/ s_m^+ which will favour the optimal projection $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ to have a value for component i closer to that in $\boldsymbol{R}_{j0}$ compared to component m. This would in turn, ceteris paribus, lead to the reflection $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ being closer to the more likely outturn levels of $\boldsymbol{R}_{j0}$. Thus the model in (15) will be assessing the efficacy of the reflection $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ whose levels reflect best the probable outturn levels of the components of $\boldsymbol{R}_{j0}$.

The weights pertaining to the deviations denoted by d^+ and d^- in (15) give expression to the varying degrees of desirability to reach the different components of the ideal target $\boldsymbol{N}^*$. The more desirable to reach a component of the target i compared to m the higher the weight attaching to the deviations d of i compared to m. Moreover, if the user is more averse to exceeding rather than falling short of the target of component i of $\boldsymbol{N}^*$ then d_i^+ would merit a higher weight than d_i^-. The converse is also true. In many instances in practice, including that of manpower planning, overshooting or undershooting a target can have cost implications. For example undershooting a target level of staff of a certain category may have significant cost implications in terms of lost output of goods or services including reputational costs until the shortfall in personnel is made good. Similarly, overshooting a personnel target will have cost implications in terms of retaining underutilised staff or redundancy payments. Such cost considerations can be estimated and then used by the DM in arriving at the weights to be used in model (15). The minimisation of the sum of weighted deviational variables d might be given higher priority than the minimisation of the weighted sum of slacks s, as the minimisation of the deviational variables d is the main instrument for identifying recruitment policies compatible with the organisation's desired manpower structure $\boldsymbol{N}^*$.

Denoting the weights of the slacks by w_i^- , w_i^+ and the weights of the deviational variables as z_i^- and z_i^+ respectively, the additive DEA- Markov manpower model is as in (15).

$$\text{Min}\left\{\sum_{i=1}^{k} w_i^- s_i^- + \sum_{i=1}^{k} w_i^+ s_i^+\right\} + \left\{\sum_{i=1}^{k}\left(z_i^- d_{ij0}^-\right) + \sum_{i=1}^{k}\left(z_i^+ d_{ij0}^+\right)\right\} \quad (15)$$

subject to

$$\sum_{j=1}^{n} \lambda_j R_{ij} + s_i^- - s_i^+ = R_{ij0} \quad \text{for } i = 1,2 \dots, k \quad (15.1)$$

$$N_i^* = \sum_{l=1}^{k} N(0)_l p_{il} + \sum_{j=1}^{n} \lambda_j R_{ij} + d_{ij0}^- - d_{ij0}^+ \quad \text{for } i = 1,2 \dots, k \quad (15.2)$$

$$\sum_{j=1}^{n} \lambda_j = 1 \quad (15.3)$$

$$\lambda_j, s_i^-, s_i^+, d_{ij0}^-, d_{ij0}^+ \geq 0.$$

The model in (15), as that in (14), is an instrument to aid the DM in arriving at policies for moving as close to the desired target $\boldsymbol{N}^*$ as possible. The weights w_i^-, w_i^+ on the slacks s and the weights z_i^- and z_i^+ on the deviational variables give the DM the ability to incorporate in the model in (15) subjective judgement both on the degree of uncertainty and the degree of desirability of attaining individual staff categories. Moreover, through varying the magnitudes of the weights for s^- relative to the weight for s^+ the DM can further express the degree of certainty when it is more likely a component might be over achieved rather than under attained, it being the case that the less likely an attainment the lower the weight pertaining to it.

The DMs specify multiple potential flows $\boldsymbol{R}_j$ which reflect historical transitions and attritions in personnel along with potential recruitments. All these are not in the gift of the DM but are generally influenced by policies adopted by the DM. Model (14) would be deployed where the DM has an equal preference over attaining individual component levels of $\boldsymbol{N}^*$ and no varying prior expectation at component level of each $\boldsymbol{R}_j$ as to the likelihood of materialising. In contrast, the model in (15) would be the one of choice where the DM has varying degrees of preference over attaining individual component levels of $\boldsymbol{N}^*$ and/or varying prior expectation at component level of each $\boldsymbol{R}_j$ as to the likelihood of materialising. In both cases model (15) offers a targeted exploration of the specified flows $\boldsymbol{R}_j$ and those created as convex combinations of the former.

Model (15), through the weighting structure over the slacks s and the deviational variables d, identifies convex combinations $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ or specified flows $\boldsymbol{R}_j$ that would be best to aim for under the preference structure concerned. Clearly if the instance of model (15) when solved yields zero values for all slacks s and deviational variables d, then $\boldsymbol{N}^*$ is one step attainable from $\boldsymbol{N}(0)$ were $\boldsymbol{R}_{jo}$ proves to be the state of nature that materialises. If any one of the (optimal) slacks s_i^-, s_i^+ or deviational variables d_{ij0}^- or d_{ij0}^+ is positive it would mean the DM would need to adopt internal policies to steer the flow $\boldsymbol{R}_{jo}$ in the direction of the flow in $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ in order to better, if not fully, attain $\boldsymbol{N}^*$. The levers in the gift of DMs to steer $\boldsymbol{R}_{jo}$ in the direction of $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ are the internal policies the organisation adopts. These may include talent management, training, job rotation, etc. (for a pertinent Markov model see, for example, Georgiou and Tsantas, 2002). In this respect, management can influence transition patterns directly or indirectly. Our approach follows an indirect path towards influencing these transitions. It is important to recall however, that the effectiveness of model (15) to lead to suitable policies for attaining $\boldsymbol{N}^*$ depends on the choice of the weights for slack and deviational variables. The weights need to reflect, as noted above, degrees of certainty over expected states of nature and degrees of desirability of attaining different components of attain $\boldsymbol{N}^*$. In addition, the weights need to reflect the scale of measurement of different components of $\boldsymbol{N}^*$. In view of the uncertainty about future states of nature and the subjective nature of degrees of desirability over the attainment of each component of N*, in practice the user could opt for sensitivity analysis to arrive at a more robust determination of the interventions that may be adopted for attaining N*. Such sensitivity analysis would involve specifying alternative sets of personnel flows $\boldsymbol{R}_j$, weights w_i^-, w_i^+ on the slacks s and weights z_i^- and z_i^+ on the deviational variables and then using the model in (15) in alternative 'what if' runs to identify, should it exist, a converging notion of the interventions needed for attaining N*.

4. A composite slack based metric of efficacy

We conclude the methodology section by noting that once model (14) or (15) is solved it is possible to compute a composite measure of the efficacy of a recruitment vector $\boldsymbol{R}_{jo}$ relative to the target $\boldsymbol{N}^*$. The measure is akin to the slack-based measures of efficiency found in the DEA literature, e.g. Tone (2001) and Cooper et al. (2007b). We propose the measure $\boldsymbol{\rho}$ in (16) which is multiplicatively decomposed. The first bracketed term to which we will refer as ρ_s, measures the distance of the flow $\boldsymbol{R}_{jo}$ under assessment from its optimal projection $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ derived from the model solved. The first term in (16) equals 1 when $\boldsymbol{R}_{jo}$ is identical with the vector $\sum_{j=1}^{n} \lambda_j^* R_{ij}$. Otherwise it exceeds 1 and the larger its value the larger the deviation between the projection $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ and $\boldsymbol{R}_{jo}$.

$$\rho = \left(1 + \frac{1}{k}\sum_{i=1}^{k} \frac{s_i^- + s_i^+}{R_{ij0}}\right) \times \left(1 + \frac{1}{k}\sum_{i=1}^{k} \frac{d_{ij0}^- + d_{ij0}^+}{N_i^*}\right) \quad (16)$$

The second bracketed term in (16), to which we will refer as ρ_d, reflects the distance of the ultimate level of personnel of each category derived through the model solved from the corresponding ideal level in $\boldsymbol{N}^*$. If the ideal target $\boldsymbol{N}^*$ were to be attained the value of the second bracketed term would be 1. Otherwise it would exceed 1 and the larger its value the larger the deviation between the ultimate level of personnel of each category derived through the model solved from the corresponding ideal level in $\boldsymbol{N}^*$.

It is noted that though under and over shoots of components s and d of each category i of staff are included in both the bracketed terms of (16), by the nature of the minimization models (14) and (15) solved, at most only one of the s and d per i can be positive at the optimal solution to either model. The aggregate measure in (16) is monotone increasing in each shortfall or overshoot and it is inspired by the Slack-Based Measure (SBM) of efficiency used in the DEA literature. The SBM measure proposed by Tone (2001) is also multiplicatively decomposed, one term reflecting input and the second term output inefficiency. These inefficiencies correspond in (16) to distances between $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ and $\boldsymbol{R}_{jo.}$ on the one hand, and between ultimate personnel levels and the ideal target $\boldsymbol{N}^*$ on the other.

Note that when the aggregate ρ equals 1 in (16), $\boldsymbol{R}_{jo.}$leads in one step to target $\boldsymbol{N}^*$. Otherwise we have ρ>1 in which case it is still possible to have attained the target $\boldsymbol{N}^*$ but only if ρ_d in (16) is 1. In this latter case, the value of ρ>1 would indicate that $\boldsymbol{R}_{jo.}$ needed adjustment to $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ before it could lead to the ideal target $\boldsymbol{N}^*$. In contrast if ρ>1 but ρ_s in (16) is 1, it would indicate that even if $\boldsymbol{R}_{jo.}$ is identical to $\sum_{j=1}^{n} \lambda_j^* R_{ij}$ it still does not lead in one step to the target $\boldsymbol{N}^*$.

A variant of the model proposed in (15) could be to replace its objective function with the measure in (16) and then solve the resulting model using nonlinear programming techniques or linear transformations based on the linear fractional programming method proposed by Charnes and Cooper (1962). In the interests of simplicity we do not pursue this avenue further. We instead recommend using the appropriate to DM preferences radial or the additive DEA-Markov model in (14) or (15) to explore alternative routes to attaining the ideal staff levels $\boldsymbol{N}^*$. Then ex post the expression in (16) can be used, if desired, to capture the degree of efficacy of each proposed recruitment vector $\boldsymbol{R}_{jo}$ for attaining the ideal staff levels in $\boldsymbol{N}^*$. This approach is amenable to the DM varying progressively the combinations of weights and specified alternative possible personnel flows in order to carry out sensitivity analysis as noted above. In view of the uncertainty inevitably surrounding both the DM preference weights and the future states of nature sensitivity analysis is a valuable tool for mitigating the impacts of uncertainty.

5. Numerical Illustrations

5.1 An illustrative application of the radial DEA-Markovian model

To illustrate the procedure of evaluating potential sets of recruitment flows using the radial hybrid DEA-Markov model in (14) we assume that we have an organization with four personnel grades, that is, k=4. The initial population vector is $\boldsymbol{N}(0) = [100 \quad 100 \quad 100 \quad 100]$. We assume that managerial aspirations look forward to a future population structure denoted by $\boldsymbol{N}^* = [300 \quad 200 \quad 150 \quad 110]$. The internal transitions probabilities have been estimated from historical mobility records (see Bartholomew et al. 1991) and are provided in the following time-homogeneous transition matrix **P** for the time horizon pertaining to the target ***N****.

$$\boldsymbol{P} = \begin{bmatrix} 0.50 & 0.25 & 0.05 & 0.00 \\ 0.00 & 0.25 & 0.50 & 0.05 \\ 0.00 & 0.00 & 0.85 & 0.10 \\ 0.00 & 0.00 & 0.00 & 0.90 \end{bmatrix}$$

The transition matrix, is upper triangular, denoting a common case in a managerial setting of negligible demotion probabilities. We further assume that the management examines the historical records of transitions, recruitment and mobility patterns and taking into account its own goals and the uncertainty surrounding future internal and external transitions it comes up with a set of seven potential recruitment flows in the sense that they capture the range of possible variations of recruitment to each level for a certain personnel prospective philosophy. The set is provided here: $\boldsymbol{R}_1 = [200 \quad 200 \quad 200 \quad 200]$, $\boldsymbol{R}_2 = [200 \quad 100 \quad 125 \quad 50]$, $\boldsymbol{R}_3 = [250 \quad 150 \quad 10 \quad 5]$, $\boldsymbol{R}_4 = [250 \quad 150 \quad 25 \quad 100]$,

$\boldsymbol{R}_5 = [150 \quad 50 \quad 400 \quad 200]$, $\boldsymbol{R}_6 = [400 \quad 150 \quad 300 \quad 200]$, $\boldsymbol{R}_7 = [250 \quad 125 \quad 150 \quad 150]$.

For illustrative purposes we have ensured that amongst the above recruitment flows there is one capable of attaining the desired structure ***N**** in a single step.

To apply the hybrid DEA-Markov model in (14) we employ lexicographic optimization by prioritising the minimization of θ, followed by the maximization of the input slacks and finally the minimization of the deviational variables. The weights we have used in the objective function appear in brackets next to the s and d variables in Table 1. The weights are consistent with assuming we have no prior information on which components of a candidate flow are more likely to materialise, and having equal preferences over the attainment of the target staff components. In the interests of simplicity we do not normalise weights of the slacks or for the deviations for the differences in magnitude between components of the targets or the staff flows being tested. We have however, assumed a higher weight for the slacks s compared to the deviational variables d consistent with the notion that the projections of the hypothesised staff flows being as realistic in terms of possible outturn values as possible before consequent distances from the targets ***N**** are assessed. Thus for example, the model for potential flow $\boldsymbol{R}_1$ is as in (17):

$$\text{Min } \theta - \varepsilon_s\left\{\sum_{i=1}^{4} s_i^-\right\} + \varepsilon_d\left\{\sum_{i=1}^{4}(d_{i1}^- + d_{i1}^+)\right\} \qquad (17)$$

Subject to

$$200\lambda_1 + 200\lambda_2 + 200\lambda_3 + 200\lambda_4 + 200\lambda_5 + 200\lambda_6 + 200\lambda_7 + s_1^- = 200\theta$$

$$200\lambda_1 + 100\lambda_2 + 150\lambda_3 + 150\lambda_4 + 50\lambda_5 + 150\lambda_6 + 125\lambda_7 + s_2^- = 200\theta$$

$$200\lambda_1 + 125\lambda_2 + 10\lambda_3 + 25\lambda_4 + 400\lambda_5 + 300\lambda_6 + 150\lambda_7 + s_3^- = 200\theta$$

$200\lambda_1 + 50\lambda_2 + 5\lambda_3 + 100\lambda_4 + 200\lambda_5 + 200\lambda_6 + 150\lambda_7 + s_4^- = 200\theta$

$$[100 \quad 100 \quad 100 \quad 100]\begin{bmatrix}0.50\\0\\0\\0\end{bmatrix} + 200\lambda_1 + 200\lambda_2 + 200\lambda_3 + 200\lambda_4 + 200\lambda_5 + 200\lambda_6 + 200\lambda_7 + d_{11}^- - d_{11}^+ = 300$$

$$[100 \quad 100 \quad 100 \quad 100]\begin{bmatrix}0.25\\0.25\\0\\0\end{bmatrix} + 200\lambda_1 + 100\lambda_2 + 150\lambda_3 + 150\lambda_4 + 50\lambda_5 + 150\lambda_6 + 125\lambda_7 + d_{21}^- - d_{21}^+ = 200$$

$$[100 \quad 100 \quad 100 \quad 100]\begin{bmatrix}0.05\\0.50\\0.85\\0\end{bmatrix} + 200\lambda_1 + 125\lambda_2 + 10\lambda_3 + 25\lambda_4 + 400\lambda_5 + 300\lambda_6 + 150\lambda_7 + d_{31}^- - d_{31}^+ = 150$$

$$[100 \quad 100 \quad 100 \quad 100]\begin{bmatrix}0\\0.05\\0.10\\0.90\end{bmatrix} + 200\lambda_1 + 50\lambda_2 + 5\lambda_3 + 100\lambda_4 + 200\lambda_5 + 200\lambda_6 + 150\lambda_7 + d_{41}^- - d_{41}^+ = 110$$

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 = 1$

$0 \le \theta, \lambda_j, s_i^-, d_{i1}^-, d_{i1}^+$ for all i and j.

Table 1, shows the results after solving the model in (17) as above, modified accordingly for each one of the seven potential recruitment flows in turn. As we can see, there is only one fully efficient recruitment flow in the strict sense, $\boldsymbol{R}_3$, as it attains $\boldsymbol{N}^*$ in one step when starting from $\boldsymbol{N}(0)$. In this case, the optimal value of λ_3 equals 1 when the instance of model (14) relates to $\boldsymbol{R}_3$. All slacks and deviational variables are zero and also the contraction factor θ has optimal value of 1 (no contraction of the levels in $\boldsymbol{R}_3$ is needed). We deduce that $\boldsymbol{R}_3$, is fully efficient. In terms of policy by the organisation the solution for $\boldsymbol{R}_3$ means no changes are required to current incentive structures, promotion, recruitment and retention decisions. Were $\boldsymbol{R}_3$ to prevail as state of nature by the end of the period for which $\boldsymbol{N}^*$ is the target the target $\boldsymbol{N}^*$ will be attained.

Please place Table 1 near here

We note that when solving (17) in respect of $\boldsymbol{R}_2$ and $\boldsymbol{R}_5$ we find their corresponding λ_2 and λ_5, have values of 1 and the radial contraction θ = 1. However, the model still yields some positive deviational variables at its optimal solution. So flows $\boldsymbol{R}_2$ and $\boldsymbol{R}_5$ are deemed "weakly efficient". We see that $\boldsymbol{R}_2$ is closer to the target $\boldsymbol{N}^*$ than $\boldsymbol{R}_5$ as the deviational variables of $\boldsymbol{R}_2$, are much lower than those of $\boldsymbol{R}_5$. Looking at the SBM measure we see that ρ=1.3981 for $\boldsymbol{R}_2$ compared to ρ=2.3 for $\boldsymbol{R}_5$, confirming the superiority of $\boldsymbol{R}_2$. The HR policy implications for the organisation are that whether $\boldsymbol{R}_2$ or $\boldsymbol{R}_5$ materialises as state of nature by the end of the planning horizon no changes in HR policies are needed but $\boldsymbol{N}^*$ itself is not likely to be attained. The optimal values of the deviational variables indicate the likely shortfalls and overshoots at each personnel grade.

Looking next at $\boldsymbol{R}_1$ and $\boldsymbol{R}_6$ we find that they have a contraction factor θ of 1 but they are not benchmark. $\boldsymbol{R}_1$ has as benchmark ('efficient peer') $\boldsymbol{R}_2$ (λ_2 = 1) and $\boldsymbol{R}_6$ has efficient peer $\boldsymbol{R}_3$ (λ_3 = 1). It is recalled that each flow such as $\boldsymbol{R}_1$ will have been specified by the user to reflect some potential 'state of nature' where recruitment is concerned (for example an optimistic view of recruiting staff of one category but pessimistic of recruiting staff of another category etc.). In view of the fact that $\boldsymbol{R}_2$ dominates $\boldsymbol{R}_1$, if the prevailing internal and external transitions ultimately point towards the flow reflected in $\boldsymbol{R}_1$ materialising, management should make internal interventions (e.g. through rewards and sanctions) to steer the transition rates going forward towards those that gave rise to flow $\boldsymbol{R}_2$. A similar statement can be made for $\boldsymbol{R}_6$ relative to $\boldsymbol{R}_3$ and it is further worth mentioning in this case that it is possible to attain the target $\boldsymbol{N}^*$ should managerial actions steer $\boldsymbol{R}_6$ to $\boldsymbol{R}_3$. This is also reflected in the ρ_d value of 1 under $\boldsymbol{R}_3$ and a ρ_s value equal to 1.5792.

Finally, we have the two "inefficient" flows $\boldsymbol{R}_4$ and $\boldsymbol{R}_7$. $\boldsymbol{R}_4$ has a radial contraction θ= 0.975 and some slacks that provide a projection which is a convex combination of $\boldsymbol{R}_2$ and $\boldsymbol{R}_3$. In this case λ_2=0.125 and λ_3=0.875. This radial projection of $\boldsymbol{R}_4$ provides a virtual recruitment vector of [243.75, 143.75, 24.375, 10.625]. This projection is close to the target $\boldsymbol{N}^*$ a fact reflected also in the close to 1 value of ρ_d=1.0498 for $\boldsymbol{R}_4$ (see Table 1). This virtual projection prompts management to modify HR policies to influence transition probabilities, recruitment and retention away from those that favour $\boldsymbol{R}_4$ and towards those that favour the materialisation of $\boldsymbol{R}_2$ and $\boldsymbol{R}_3$ and especially $\boldsymbol{R}_3$, in view of the latter dominating the convex target with λ_3=0.875. *Target virtual flows of this type illustrate an important contribution of DEA within the Markovian manpower planning model. They enable the user to identify potential recruitment flows, not initially specified, which offer a better path to the ideal manpower structure thus increasing the probability of attainability*. An analogous interpretation of the solution for flow $\boldsymbol{R}_4$ can be made also for $\boldsymbol{R}_7$. However, as can be gathered from Table 1, in this case HR policies should aim towards those favouring $\boldsymbol{R}_2$ and even then the target $\boldsymbol{N}^*$ will not be likely to be reached as can be seen from the optimal deviation values for $\boldsymbol{R}_7$ in Table 1 and the rather large value of the ρ_d = 1.3867.

In concluding our look at Table 1 we note the component ρ_d of the SBM measure reflects the optimal projection to the boundary of the DEA-Markovian model of the corresponding initially specified potential staff flow. This brings to the fore one of the key features of DEA that we exploit within the MCs framework: that is, through the projections of the initial manpower flows the model identifies virtual flows which are more efficacious in leading towards the target vector $\boldsymbol{N}^*$ than the originally specified 'best guess' flows.

5.2 An illustrative application of the additive DEA-Markovian model

We illustrate the additive DEA-Markovian model using data from the paper De-Feyter and Guerry (2009) (DFG). It is important to underline, that the Markovian framework of the DFG paper is set in an exclusively stochastic environment based on fuzzy set theory while our approach, in essence, simulates the stochastic context within a deterministic approach. Nevertheless, since the DFG paper uses expected staff flows in the same way we use them in this paper, it lends itself for contrasting the two approaches. The DFG paper assumes 3 categories of staff, an initial population vector of $\boldsymbol{N}(0) = [200 \quad 275 \quad 225]$ and a target vector $\boldsymbol{N}^* = [200 \quad 260 \quad 230]$. The authors provide data of transitions and using maximum likelihood estimators they calculate the following transition matrix **P**.

$$\boldsymbol{P} = \begin{bmatrix} 0.791 & 0.102 & 0.056 \\ 0.062 & 0.739 & 0.102 \\ 0.049 & 0.049 & 0.802 \end{bmatrix}$$

As it is clear from the above, demotions are allowed, albeit with very low probabilities. The set of 5 potential recruitment vectors is $\boldsymbol{R}_1 = [14 \quad 27 \quad 10]$, $\boldsymbol{R}_2 = [13 \quad 24 \quad 9]$, $\boldsymbol{R}_3 = [14 \quad 28 \quad 10]$, $\boldsymbol{R}_4 = [20 \quad 19 \quad 13]$ and $\boldsymbol{R}_5 = [8 \quad 36 \quad 7]$. We know that none of these recruitment flows is capable of attaining the desired structure $\boldsymbol{N}^*$

in a single step. The instance of model (15) populated with the DFG data and set to assess the efficacy of potential flow $\boldsymbol{R}_5$ is as in (18):

$$\text{Min}\left\{\sum_{i=1}^{3} w_i^- s_i^- + \sum_{i=1}^{3} w_i^+ s_i^+\right\} + \left\{\sum_{i=1}^{3}(z_i^- d_{i5}^-) + \sum_{i=1}^{3}(z_i^+ d_{i5}^+)\right\} \quad (18)$$

Subject to

$$14\lambda_1 + 13\lambda_2 + 14\lambda_3 + 20\lambda_4 + 8\lambda_5 + s_1^- - s_1^+ = 8$$

$$27\lambda_1 + 24\lambda_2 + 28\lambda_3 + 19\lambda_4 + 36\lambda_5 + s_2^- - s_2^+ = 36$$

$$10\lambda_1 + 9\lambda_2 + 10\lambda_3 + 13\lambda_4 + 7\lambda_5 + s_3^- - s_3^+ = 7$$

$$[200 \quad 275 \quad 225]\begin{bmatrix}0.791\\0.062\\0.049\end{bmatrix} + 14\lambda_1 + 13\lambda_2 + 14\lambda_3 + 20\lambda_4 + 8\lambda_5 + d_{15}^- - d_{15}^+ = 200$$

$$[200 \quad 275 \quad 225]\begin{bmatrix}0.102\\0.739\\0.049\end{bmatrix} + 27\lambda_1 + 24\lambda_2 + 28\lambda_3 + 19\lambda_4 + 36\lambda_5 + d_{25}^- - d_{25}^+ = 260$$

$$[200 \quad 275 \quad 225]\begin{bmatrix}0.056\\0.102\\0.802\end{bmatrix} + 10\lambda_1 + 9\lambda_2 + 10\lambda_3 + 13\lambda_4 + 7\lambda_5 + d_{35}^- - d_{35}^+ = 230$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1$$

$\lambda_j, s_i^-, s_i^+, d_{i5}^-, d_{i5}^+ \geq 0$ for all *i* and *j*,

where $w_i^+, w_i^-, z_i^+, z_i^-$ are user specified weights.

The weights w_i^+, w_i^- on the slacks s are intended to drive the optimal reflection of the components of the flow being assessed, $\boldsymbol{R}_5$ in model (18), closer to what the DM expects their levels to turn out to be when we reach the end of the planning horizon. For example, if we expect the outturn value of component i to exceed rather than fall short of its specified value within $\boldsymbol{R}_5$ then a higher weight w_i^- compared to w_i^+ would be assigned. Within the minimisation objective function of model (18) this will favour a higher value s_i^+ compared to s_i^- in line with DM expectations. It is recalled that only one of s_i^+ compared to s_i^- can have a positive value at the optimal solution to Model (18). The weights z_i^+, z_i^- on the deviational variables d are intended to reflect DM relative preferences between exceeding or falling short of the target value of component i. For example, in the event the target value for component i in N* cannot be met exactly, if the DM prefers to undershoot rather than overshoot it then the weight z_i^+ should be higher than the weight z_i^-.

For those cases where the magnitudes of components are different in terms of scale, the weights further need to be normalised, or alternatively the components of the flows ***R*** within the model in (18) need to be normalised, to eliminate implicit weights contingent on differences in component magnitudes. In the interests of simplicity and in view of the relatively similar scale size of staff components in flows $\boldsymbol{R}_5$ and targets in ***N**** in our illustrative data, we ignore here the normalisation for scale size. Finally, though we have applied the same set of weights when solving model (18) for each flow ***R*** in turn, this need not be the case in practice, especially when it comes to the weights on slacks s_i^- or s_i^+ where the degree of confidence may vary both within components and across recruitment flows.

We begin by solving an instance of the additive model in (18) in respect of each one of the flows $\boldsymbol{R}_1$ to $\boldsymbol{R}_5$ in turn. We have used the weights appearing in brackets next to the slacks s and deviational variables d in Table 2. For example, for s_1^- (denoted s1m) the weight is 1 while for s_1^+ (denoted s1p) the weight is 10. The weights for the deviation d1m and d1p follow a similar pattern. The weights used imply the DM expects for each component the outturn value is more likely to be below rather than above its specified value within the flow. Further, where necessary, the DM prefers target component values within $\boldsymbol{N}^*$ to be under rather than overshot.

It is noteworthy that with few exceptions the slacks and deviational variables take values in line with the weights intended to reflect DM expectations and preferences. Thus for $\boldsymbol{R}_1$-$\boldsymbol{R}_4$ the slacks s1m, s2m and s3m all have positive optimal values while the corresponding s(i)p values are zero, except s2p of $\boldsymbol{R}_4$ which is positive. In other words, the optimal projections for $\boldsymbol{R}_1$-$\boldsymbol{R}_4$ incorporate overwhelmingly undershoots from the values specified in these flows. The practical implication of this is that the model uses projections of the flows $\boldsymbol{R}$ as close to expectation as possible. This in turn means the evaluation of the efficacy of the flows specified by the DM for getting to the target $\boldsymbol{N}^*$ is as close to the DM expectations of outturn values as possible.

Flow $\boldsymbol{R}_5$ bucks the trend in that despite the DM expectations, the model produces a reflection with positive values for s1p and s3p suggesting it is preferable to plan for overshoots of these two components of this flow. The practical implication of this outcome for $\boldsymbol{R}_5$ is that the DM should aim, to the extent it might be possible, to influence (for example through internal promotions and retentions) to overshoot these two components for a better chance of getting closer to the targets in $\boldsymbol{N}^*$. It is perhaps worth noting that components 1 and 3 have very low values in $\boldsymbol{R}_5$ and so overshooting them maybe desirable even if difficult or unlikely.

The ρ_s index values in Table 2 reflect how close the optimal projected values of each flow $\boldsymbol{R}$ are to its original specification. In the case of $\boldsymbol{R}_2$ the value is 1 and so the optimal projection and the specified value coincide. At the other extreme the ρ_s value for $\boldsymbol{R}_5$ at 1.368, is the highest suggesting this flow has a specified value the furthest from expectation.

As far as deviation values are concerned undershoot values d1m, d2m and d3m are the only ones to take positive values, in line with the weights used for deviational variables in model (18). The ρ_d index values are all close to 1 suggesting the projections of the original flows, if attained, would all get us close to the target $\boldsymbol{N}^*$. Thus the key value of the model is in revealing projections of flows which are efficacious in reaching $\boldsymbol{N}^*$. Of these the most efficacious is $\boldsymbol{R}_2$ while $\boldsymbol{R}_1$ and $\boldsymbol{R}_3$ are the next best in terms of the ρ_s and indeed SBM overall values.

It is worth pointing out that these results are in line with the results in the DFG paper even though the modelling context is different. DFG find that in a deterministic framework of their approach $\boldsymbol{R}_2$ is more efficacious while using a stochastic approach $\boldsymbol{R}_1$ is preferred. In our approach, as noted above, both $\boldsymbol{R}_2$ and $\boldsymbol{R}_1$ are among the most preferred flows, notwithstanding the fact that our approach relies on a weighting structure in the objective function of model (18) to capture the stochastic information used in the DFG paper.

To further illustrate the working of the additive model we now focus on flow $\boldsymbol{R}_5$ which in DFG, as well as in our model above, was ranked at the bottom of the preference list. Let us assume the probability of achieving at the end of the horizon overshoots of inflows of 8 and 7 in staff grades 1 and 3 respectively, is rather high and certainly higher than the probability of realisation of undershoots of these grades. This is not an unreasonable assumption if we assume that the low values of 8 and 7 do signal some confidence that these levels as a minimum have been secured. By a similar reasoning we can assume that overshooting of grade 2 is less likely than undershooting it, given the relatively high level specified for that component. Table 3 shows the weighting structure adopted to reflect these expectations. The relative values of the weights can be seen in parentheses next to the variable names. Thus the weight on s1m and s3m is 100 compared to 1 for s1p and s3p respectively. In contrast the weight is 100 for s2p and 1 for s1m. Regarding the achievement of targets in ***N**** we have assumed in this variation that over achievement of staff target numbers is preferred rather than under achievement and so we have used larger weights for all d_{ij}^{-}.

The solution of this model is presented in Table 3. As can be seen the model yields a solution where indeed the slacks s1m and s3m are zero and s1p and s3p are positive. In contrast, s2m is positive and s2p is zero. So the optimal slack values are in line with expectation. However, the resulting optimal reflection is quite far from the specified flow $\boldsymbol{R}_5$ as ρ_s = 1.5146 in Table 3 compared to ρ_s =1.368 in Table 2 indicates. On the positive side, this reflection is marginally closer to the ideal values in ***N**** as the ρ_d = 1.0037 in Table 3 compared to ρ_d = 1.0044 in Table 2 shows. So the reflection of $\boldsymbol{R}_5$ depicted in Table 3 is marginally better to aim for rather than that in Table 2. Note in this respect that the reflection of $\boldsymbol{R}_5$ is in large measure $\boldsymbol{R}_3$ as deduced from λ_3=0.9 in Table 3. This further signifies how the weights reflecting a combination of likelihood and preferences judgements by the DM alter the actions the DM should take in order to get as close as possible to the ultimate target ***N****.

We continue our comparison of the additive deterministic DEA model with the approach in DFG by adding to the five staff flows above, a 6th flow $\boldsymbol{R}_6$ = [14 25 10] which DFG present as one capable of attaining in one step the target ***N**** = [200 260 230] when starting from ***N***(0) = [200 275 225]. Apart from introducing this 6th 'desirable' flow we also incorporate differences in weights as might be the case when the DM is much more certain about the likely future value of one component in particular for the potential flows specified. This for example might be the case when for one potential staff category the DM may have a substantial degree of control, e.g. some level of current staff for which supply is plentiful. The weights we have used in this illustration appear next to the slacks s and deviational variables d in Table 4. The weights indicate that the DM is far more certain of an overshoot occurring in flow component 1 than is the case for undershoots or overshoots in components 2 and 3. Having said that, the DM is more certain of undershoots than overshoots in components 2 and 3, albeit not to the degree of certainty regarding component 1. The solution of model (18) modified to include $\boldsymbol{R}_6$ and the foregoing weights can be found in Table 4.

Table 4 shows that in response all optimal reflections of the specified flows have at least the specified value of component 1 as can be seen from the zero s_1^- values across all flows ***R***. This leads to re-adjustments of the optimal projections for the other two components across all flows ***R*** and reflected in the fact that the ρ_s values in Table 4 are the same or lower than the corresponding values in Table 2. The improved ρ_s values in Table 4 compared to Table 2 suggest projected flows are on the whole closer to the originally specified flows which can be seen as a consequence of the DM having given the model a stronger indication of the likely outturn levels of staff going forward. It is noteworthy also that the overall SBM values in Table 4 are the same or closer to 1 than those in

Table 2. This is as we might expect given the prevailing improvement in the ρ_s values from Table 2 to Table 4. It is also noteworthy that in Table 4 the preferred flow is $\boldsymbol{R}_6$ just as in DFG. This is concluded both from the closest overall SBM value of almost 1 in Table 4 and the fact that λ_6 = 0.91 at the optimal solution to the instance of model (18) corresponding to $\boldsymbol{R}_6$. This makes $\boldsymbol{R}_6$ by far the most dominant influence on its optimal projection. In fact $\boldsymbol{R}_6$ is also the dominant influence in the optimal projections of the flows $\boldsymbol{R}_1$ and $\boldsymbol{R}_3$ as can be deduced from the value of 0.88 for λ_6 for these two flows in Table 4. Finally, a further interesting impact of the stronger information on likely outcome regarding staff category 1 compared to the rest can be seen when we compare the outcome for flow $\boldsymbol{R}_4$ in Tables 2 and 4. Flow $\boldsymbol{R}_4$ is one of the weaker flows in Table 2 with overall ρ=1.2798. It has only marginal impact on its optimal projection with λ_4=0.10537. Yet in Table 4 its ρ index drops to 1.0225 making it one of the stronger flows with its projection being flow $\boldsymbol{R}_4$ itself as we have λ_4 = 1.

Clearly the results of the additive model are data dependent and contingent on the weighting structure used on slacks and deviational variables. Therein, however, also lies the strength of our approach. It offers the DM the means to explore, using sensitivity analysis, the decision space by bringing to bear a full range of potential courses of action, in combination with alternative levels of perceived certainty of likely outturn values all assessed relative to her underlying preferences over ultimate target outcomes.

Please place Tables 2, 3 and 4 near here

5.3 A schematic representation of the DEA-Markov approach

We conclude the illustration of our DEA –Markov decision aid with the schematic representation in Figure 1 of its main steps for the generic case.

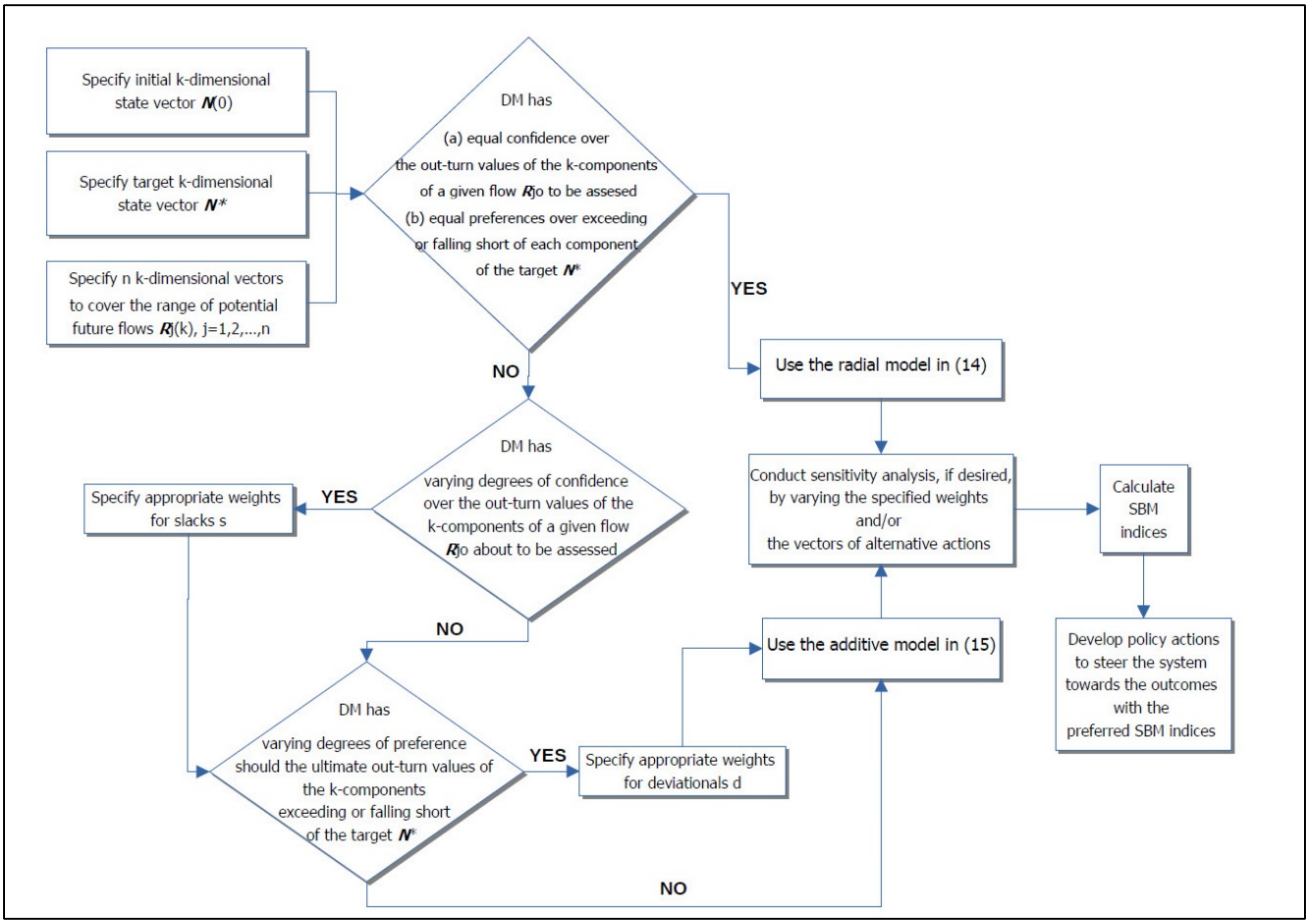

Figure 1. Schematic representation of the main steps

6. Conclusion

The main contribution of this paper is a DEA-enabled approach to aid decision making in contexts hitherto addressed using Markov Chains. The use of DEA makes it possible to circumvent complexities caused by the inherent stochastic nature of the problem addressed, through the use of a combination of user-specified potential alternatives and weights reflecting uncertainty and preferences over final outcomes. This approach will make easier the use of MC models in a variety of contexts.

To present our methodology, without loss of generality, we have used a classic Markov manpower planning model. The approach assumes an organisation which has a current personnel structure and wishes to attain an ideal structure within a given planning horizon. The method developed is used for generating and evaluating potential recruitment policies for reaching as close as possible the desired personnel structure. The approach begins with the Decision Maker specifying potential personnel flows over the planning horizon based on historical data and informed judgements of future recruitment and transition possibilities between personnel categories. The DEA assumption of convexity is then used to form an infinite set of feasible courses of action (recruitment flows). Then, still within the DEA framework, the efficacy of each initially specified course of action and of its optimal reflection within the infinite set of actions, is measured for attaining the ultimate state of the system under investigation. The most suitable of these virtual flows are identified as potential candidates that management can attempt to bring about, as far as possible, through instruments at its disposal within the organisation. Using weights in a goal programming objective function our hybrid DEA-Markov method can reflect the Decision Maker's subjective confidence about the realization of specific policies or staff flow levels as well as the desirability and importance of specific targets on staff categories. In essence, the proposed framework offers a means of converting an otherwise stochastic problem into a deterministic one, by offering the opportunity to the Decision Maker to express their attitudes on likelihoods and preferences in a tractable manner.

Our approach, presented here in the context of Markovian manpower planning, can be readily repurposed to aid other decision contexts. For example, a field where Markov chains are also used is that of health care delivery. Our approach can be used to identify the most efficacious way from the cost perspective of managing a chronic illness such as diabetes. In this case flows and transitions are affected by treatment protocols and healthcare interventions informed by medical research and expected patterns of mobility due to new drugs or healthcare policies (e.g. home treatment vs hospitalization). Models of the type we have proposed here, once developed to the more complex area where both clinical outcomes, alternative clinical interventions and quality of life are relevant, could aid health professionals in managing chronic diseases. Although Markov processes have long since been used to model mobility patterns in chronic diseases, we are not aware of any approaches comparing the efficiency of alternative policies using approaches like DEA.

We have used in this paper the classical DEA model in the context of Markov chains. However, further research is possible for additional contributions to Markov chain models that may be possible through extended DEA models. In particular, DEA models for estimating allocative efficiencies could be usefully explored. Allocative efficiencies take into account prices of the factors for delivering goods or services and reflect gains that could be made by aligning the mix of resources used with their prices. Another type of DEA model that could be explored are network models. In network DEA models, outcomes from one stage feed into subsequent stages before final outcomes are delivered. For example, we can expand the one step attainability problem we have considered into the multiple step attainability problem. For instance, in the manpower planning context the manpower structure resulting in a given step forms the starting manpower structure for the next stage and so on. This format is the type of problem handled in network DEA models. In addition, the possibility of using the transition probabilities as control variables could be explored further. This approach would give the decision maker an additional lever to drive the system towards desirable structures, e.g. manpower or health outcomes, as derived through the solution of the DEA models involved. This more direct approach is an important aspect, especially in manpower planning where it has been occasionally used in Markov modeling, with relative caution though, due its increased computational complexity.

In summary, our approach has opened an avenue for exploring how DEA models can render the handling of Markov chain models more tractable. This paper has used only the basic DEA models. Further research can explore benefits that may be available from using extended DEA models.

Acknowledgment

The authors gratefully acknowledge the comments and suggestions of the *three reviewers* that improved significantly both the content and the presentation of the paper

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Table 1: Example 1. Lambdas, slacks, deviational variables, radial θ and SBM indices for seven recruitment potentials – radial model

	λ1	**λ2**	**λ3**	**λ4**	**λ5**	**λ6**	**λ7**	**S1m** (-0,001)	**S2m** (-0,001)	**S3m** (-0,001)	**S4m** (-0,001)	**d1m*** (0,0001)	**d1p** (0,0001)	**d2m** (0,0001)	**d2p** (0,0001)	**d3m** (0,0001)	**d3p** (0,0001)	**d4m** (0,0001)	**d4p** (0,0001)	**θ**	**ρ_s**	**ρ_d**	**$\rho = \rho_s + \rho_d$**
R1		1							100	75	150	50		50			115		45	1	1.4063	1.3981	1.9661
R2		1										50		50			115		45	1	1	1.3981	1.3981
R3			1																	1	1	1	1
R4		0.125	0.875						2.5		86.875	6.25		6.25			14.375		5.625	0.975	1.2214	1.0498	1.2821
R5					1							100		100			390		195	1	1	2.3015	2.3015
R6			1					150		290	195									1	1.5792	1	1.5792
R7		0.97143	0.02857					1.428			73	48.571		48.571			111.714		43.714	0.811	1.1231	1.3867	1.5574

* dim or dip represent the deviational variables d_{ij0}^{-} and d_{ij0}^{+} of the corresponding potential recruitment vector

Table 2: Example 2. Results for Flows R1-R5 - additive model in (18)

	λ1	**λ2**	**λ3**	**λ4**	**λ5**	**S1m** (1)	**S1p** (10)	**S2m** (1)	**S2p** (10)	**S3m** (1)	**S3p** (10)	**d1m** (1)	**d1p** (10)	**d2m** (1)	**d2p** (10)	**d3m** (1)	**d3p** (10)	**ρ_s**	**ρ_d**	**$\rho = \rho_s + \rho_d$**
R1	0.50288	0.46538		0.03173		0.275		1.65		0.37019						0.67019		1.0393	1.0010	1.0403
R2		1										0.725		1.35		1.3		1	1.0048	1.0048
R3	0.50288	0.46538		0.03173		0.275		2.65		0.37019						0.67019		1.0504	1.0010	1.0515
R4		0.89643		0.10537		6.275			4,48214	3.58571				1.86786		0.88571		1.2752	1.0037	1.2798
R5		0.88750			0.11250		4.4375	10.65			1.775	1.2875				1.52500		1.3680	1.0044	1.3740

Table 3: Example 2. Results for ***R*₅** – additive model an assortment of weights reflecting alternative likelihoods and desires

	λ1	**λ2**	**λ3**	**λ4**	**λ5**	**S1m** (100)	**S1p** (1)	**S2m** (1)	**S2p** (100)	**S3m** (100)	**S3p** (1)	**d1m** (100)	**d1p** (1)	**d2m** (100)	**d2p** (1)	**d3m** (100)	**d3p** (1)	**ρ_s**	**ρ_d**	**$\rho = \rho_s + \rho_d$**
R5			0.9	0.1			6.6	8.9			3.3		0.875		1.75			1.5146	1.0037	1.5202

Table 4: Example 2. Lambdas, slacks, deviational variables, and SBM indices for six recruitment potentials – additive model uncertainties differ across components

	λ1	**λ2**	**λ3**	**λ4**	**λ5**	**λ6**	**S1m** (100)	**S1p** (1)	**S2m** (1)	**S2p** (10)	**S3m** (1)	**S3p** (10)	**d1m** (1)	**d1p** (10)	**d2m** (1)	**d2p** (10)	**d3m** (1)	**d3p** (10)	**ρ_s**	**ρ_d**	**$\rho = \rho_s + \rho_d$**
R1			0,11667			0.88333			1,65					0,27500			0,30000		1.0204	1.0009	1.0213
R2		1											0.725		1.35		1.30000		1	1.0048	1.0048
R3			0,11667			0,88333			2,65					0,27500			0,30000		1.0315	1.0009	1.0325
R4				1										6,27500	6,35000			2,7	1	1.0225	1.0225
R5		0.88750			0.11250			4.4375	10.65			1.775	1.2875				1.52500		1.3680	1.0044	1.3740
R6		0.05441			0.03676	0.90882											0.46471		1	1.0007	1.0007