An Alternative Ranking of DMUs Performance for the ZSG-DEA Model

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ABSTRACT: The Zero-Sum Gains Data Envelopment Analysis (ZSG-DEA) model was developed to evaluate the performance of Decision Making Units (DMUs) under output interdependency, which is evident when the total (over DMUs) observed output is *a priori* fixed. In this paper, we take a closer look at the derivation and interpretation of the ZSG-DEA efficiency scores and we explain why these are not really comparable across DMUs. Moreover, we verify that DMUs' ranking based on their ZSG-DEA efficiency scores is in fact incompatible with output interdependency, which requires total potential output of DMUs to be equal to their total observed output. Then, we propose an alternative metric of DMUs' performance that is both consistent with output interdependency and comparable across DMUs. This metric is used for classifying DMUs into three (high, average or low performance) groups and for ranking them within these groups. To illustrate its empirical applicability, we use data from the Olympic Games, where output interdependency is clearly evident since the total number of awarded (gold, silver, and bronze) medals is *a priori* fixed.

KEYWORDS: Data Envelopment Analysis (DEA); Output Interdependency; Fixed-Sum Output; Zero-Sum Gains (ZSG); Performance Metric

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1. Introduction

The aim of the Zero-Sum Gains Data Envelopment Analysis (ZSG-DEA) model, developed by Lins *et al.* (2003), is to evaluate the performance of Decision Making Units (DMUs) under output interdependency, i.e., when the output level of each DMU depends on the output levels of all other DMUs; see Table 1 for a brief literature review of the topic. This is the case when the total (over DMUs) observed output is *a priori* fixed, as for example, (*i*) if firms' market share (Hu and Fang, 2010) or the volume of distributed bank loans (Amirteimoori *et al.*, 2017) is used as the relevant output variable, (*ii*) when examining quota trading in the case of a fixed-sum undesirable output, such as greenhouse emissions (Gomes and Lins, 2008), and (*iii*) in sports where success for an athlete or a team is inevitably connected with all other participating athletes or teams (see Lins *et al.* (2003) for the Olympic Games, Collier *et al.* (2011) for American football, and Bi *et al.* (2015) for European football). In these cases, output interdependency affects the efficiency scores of DMUs by implying that their total potential output should be equal to their total observed output. If however output interdependency is not accounted for, then the total potential output of DMUs is proved to be greater than their total observed output and thus, the evaluated DMUs appear as worse performers than they actually are (Lins *et al.*, 2003).

The ZSG-DEA model attempts to address this problem by ensuring that the additional output, which each evaluated DMU might require to reach the efficient frontier, is gained from all other DMUs, whose output losses can be formulated by means of three different output reduction strategies, i.e., the proportional (see Lins *et al.* (2003)), the equal (see Lins *et al.* (2003) and Collier *et al.* (2011)), and the minimal (see Yang *et al.* (2011)). In the proportional reduction strategy, output losses are proportional to DMUs' observed outputs; in the equal reduction strategy, they are the same for all DMUs; and in the minimal reduction strategy, they are the minimum possible for all DMUs.

Regardless of the output reduction strategy, the ZSG-DEA efficiency score of each DMU reflects its achievements under the assumption that only this DMU, considered to be some kind of a “leader”, can gain additional output from all other DMUs, considered to be some kind of “followers”, without at the same time losing any of its output from them. By accounting for the aforementioned output gain and output losses, one can determine both the potential output of the evaluated DMU and the resultant outputs of all other DMUs, whose sum is *always* equal to the total (over DMUs) observed output implying that output interdependency is taken into consideration in each *separate* “run” of the ZSG-DEA model.[1] However, in the estimation of the ZSG-DEA efficiency scores, only the different output gains of DMUs are taken into account and for this reason, these performance measures might be characterized as normative in the sense of showing what the efficiency of each DMU could be, if this was the “leader” and each other DMU a “follower”.

From the interpretation of these efficiency scores, it is clear that performance evaluation by means of the ZSG-DEA model is based on different efficient frontiers in contrast to the assessment made by the conventional DEA model, where all DMUs are evaluated relative to a common efficient frontier. In fact, the ZSG-DEA model may estimate as many efficient frontiers as the number of the evaluated DMUs. As Gomes and Lins (2008, p. 617) put it: “the way one DMU reaches its target in the efficient frontier implies changing the frontier”. As a result, “DMUs are finally evaluated based on different efficient frontiers” (Wu *et al.*, 2019, p. 733) or more precisely, all ZSG-DEA inefficient DMUs that require additional output are evaluated based on different frontiers while all ZSG-DEA efficient DMUs, whose output gains are zero, are evaluated relative to a common frontier that is actually the conventional DEA frontier. Consequently, DMUs’ ranking based on their ZSG-DEA efficiency scores can be misleading as these performance measures, computed relative to different efficient frontiers, are not really comparable across DMUs. In addition, if one attempts to measure the total potential output of DMUs based on their ZSG-DEA efficiency scores, he/she will find that this sum is greater than DMUs’ total observed output, which implies that output interdependency is in fact not accounted for.

The first attempt to deal with these problems was made by Gomes and Lins (2008), who proposed an approach, also followed by Feng *et al.* (2019) and Liu *et al.* (2021), which is consistent with output interdependency since it permits all ZSG-DEA inefficient DMUs as a group to gain additional output from the ZSG-DEA efficient ones in order to reach the Uniform Maximum Efficiency Frontier (UMEF). Some years later, the Equilibrium Efficiency Frontier (EEF) approach was introduced and gradually developed by Yang *et al.* (2014, 2015, 2021), Zhu *et al.* (2020b, 2021), Wu *et al.* (2019), Amirteimoori *et al.* (2020), Mohamadinejad *et al.* (2021), and Li *et al.* (2021a, 2021b) that also tried to deal with the aforementioned problems of the ZSG-DEA model. According to this approach, the EEF is derived from the conventional DEA frontier in one or in several steps and it is not always unique. This is important since the existence of multiple EEFs causes "major differences in DMUs' efficiencies and rankings resulting from these frontiers" (Chen *et al.*, 2021, p. 238). For this reason, Fang (2016) and Zhu *et al.* (2017, 2020a) tried to overcome this limitation by using a secondary goal model to obtain a unique EEF.

In this paper, we take a different route to address the aforementioned problems of the ZSG-DEA model. Specifically, we follow Bernardo *et al.* (2020) and, instead of trying to create *any* frontier, we use for each DMU both its potential output, determined when the specific DMU is considered as the "leader", and its resultant outputs, determined when the specific DMU is considered as a "follower", to form a ratio of their average to the observed output of the specific DMU. In this way, we develop an alternative metric of DMUs' performance that is easily computed, comparable across DMUs since it considers all the estimated output gains, output losses and resultant outputs, and consistent with output interdependency. This metric is not an efficiency score *per se* and for this reason, it is not directly comparable to the efficiency scores obtained from the UMEF or the EEF. However, it can be used to group DMUs as high, average or low performers and rank them within each group. Thus, it seems that our approach can apply quantitative analysis to several interdisciplinary decision problems arising in the area of socio-economic planning and development and in the service and public sectors. To illustrate its applicability, we use data from the 2000 Olympic Games and then, we compare and contrast the resulting DMUs' rankings with those based on their ZSG-DEA efficiency scores.

2. Methods and Materials

Lins *et al.* (2003) developed the output-oriented Variable Returns-to-Scale (VRS) ZSG-DEA model as:

$$\begin{array}{lll} \underset{h^k,\lambda_k^i,\tilde{y}_{kr}^i}{Max} & h^k & \\ s.t. & \sum_{i=1}^{I}\lambda_k^i x_m^i \leq x_m^k, & m=1,\dots,M; \\ & \sum_{i=1}^{I}\lambda_k^i \tilde{y}_{kr}^i \geq h^k y_r^k, & r=1,\dots,R; \\ & \sum_{i=1}^{I}\lambda_k^i = 1, & \\ & \lambda_k^i \geq 0, & i=1,\dots,k,\dots,I; \end{array} \tag{1}$$

where x refers to the inputs, y to the observed outputs, whose sum over DMUs is *a priori* fixed, λ to the intensity variables, h to the ZSG-DEA efficiency score that is greater-than-or-equal-to one, $\tilde{y}$ to the resultant outputs, m is employed to index inputs, r to index outputs, and i to index DMUs. The main difference between the above model and the relevant conventional DEA model is that, except for λ and h, $\tilde{y}$ is also a decision variable in (1). Equivalently, the ZSG-DEA model in (1) may be written as:

$$\begin{array}{lll} \underset{h^k,\lambda_k^i,l_{kr}^i,z_r^k}{Max} & h^k & \\ s.t. & \sum_{i\neq k}^{I}\lambda_k^i x_m^i + \lambda_k^k x_m^k \leq x_m^k, & m=1,\dots,M; \\ & \sum_{i\neq k}^{I}\lambda_k^i\left(y_r^i - l_{kr}^i\right) + \lambda_k^k(y_r^k + z_r^k) \geq h^k y_r^k, & r=1,\dots,R; \\ & \sum_{i=1}^{I}\lambda_k^i = 1, & \\ & \lambda_k^i \geq 0, & i=1,\dots,k,\dots,I; \end{array} \tag{2}$$

where $y^k + z^k = \tilde{y}_k^k = h^k y^k$ and $y^i - l_k^i = \tilde{y}_k^i$ with $z^k \geq 0$ being the output gain of the evaluated DMU, which is considered as the "leader", and $l_k^i \geq 0$ the output loss of each other DMU that is considered as a "follower". It should be stressed here that the resultant output $\left(\tilde{y}_k^k\right)$ of the evaluated DMU is actually its potential output and that, as in (1), the formulation in (2) is non-linear and this restricts the applicability of the ZSG-DEA model.

However, Lins *et al.* (2003) have shown that this can be overcome by employing a single output $(r = 1)$ and a specific output reduction strategy.[2] Note in particular that, in the case of the proportional reduction strategy, the output gain of the evaluated DMU is given as $z^k = (h^k - 1)y^k$ and the output loss of each other DMU as $l_k^i = y^i z^k / \sum_{i\neq k}^{I} y^i$ indicating its proportionality to the observed output. By substituting these into (2) and given that $r = 1$, we obtain:

$$
\begin{array}{lll}
\underset{h^k, \lambda_k^i}{Max} & h^k & \\
s.t. & \sum_{i \neq k}^{I} \lambda_k^i x_m^i + \lambda_k^k x_m^k \leq x_m^k\,, & m = 1, \dots, M; \\
& \sum_{i \neq k}^{I} \lambda_k^i y^i RC_k + \lambda_k^k h^k y^k \geq h^k y^k, & \\
& \sum_{i=1}^{I} \lambda_k^i = 1, & \\
& \lambda_k^i \geq 0, & i = 1, \dots, k, \dots, I;
\end{array} \tag{3}
$$

where $RC_k = 1 - \left((h^k - 1)y^k / \sum_{i \neq k}^{I} y^i\right)$ is the reduction coefficient in the k^{th} "run" of (3), with $0 < RC_k \leq 1$. On the other hand, in the case of the equal reduction strategy, the output gain of the evaluated DMU is again given as $z^k = (h^k - 1)y^k$ whereas the output loss of each other DMU as $l_k^i = z^k/(I - 1)$ indicating that, being independent of the observed output, it is common for all the "followers". By substituting these into (2) and given that $r = 1$, we obtain:

$$
\begin{array}{lll}
\underset{h^k, \lambda_k^i}{Max} & h^k & \\
s.t. & \sum_{i \neq k}^{I} \lambda_k^i x_m^i + \lambda_k^k x_m^k \leq x_m^k\,, & m = 1, \dots, M; \\
& \sum_{i \neq k}^{I} \lambda_k^i \left(y^i - o_k\right) + \lambda_k^k h^k y^k \geq h^k y^k, & \\
& \sum_{i=1}^{I} \lambda_k^i = 1, & \\
& \lambda_k^i \geq 0, & i = 1, \dots, k, \dots, I;
\end{array} \tag{4}
$$

where $o_k = (h^k - 1)y^k/(I - 1)$ is the output loss in the k^{th} "run" of (4).

More importantly, Lins *et al.* (2003), Bi *et al.* (2014), and Bouzidis and Karagiannis (2021b) have shown that there is even no need for solving (3) or (4) as the ZSG-DEA efficiency scores can be computed directly from the conventional DEA efficiency scores. In particular, according to Lins *et al.* (2003) Target's Assessment Theorem (TAT), [3]

$$
h^k = \theta^k RC_k \tag{5a}
$$

where θ refers to the conventional DEA efficiency score that is greater-than-or-equal-to one, or equivalently (Bi *et al.*, 2014; Bouzidis and Karagiannis, 2021a):

$$h^k = \frac{\theta^k \sum_{i=1}^{I} y^i}{\sum_{i \neq k}^{I} y^i + \theta^k y^k} \tag{5b}$$

On the other hand, according to Bouzidis and Karagiannis (2021b):

$$h^k = \theta^k - (o_k / y^k) \tag{6a}$$

or equivalently (Bi *et al.*, 2014):

$$h^k = \frac{\theta^k (I-1) + 1}{I} \tag{6b}$$

Based on the computed ZSG-DEA efficiency scores, one can then estimate DMUs' output gains, output losses, and resultant outputs that provide alternative useful information about their performance.

Nevertheless, previous studies using the ZSG-DEA model rather ignored these and based their performance evaluations and rankings of DMUs solely on the obtained efficiency scores that, as we have noticed, are kind of normative performance measures reflecting the achievements of DMUs under the assumption that each one of them is the "leader" at the time and thus, able to gain additional output, whereas none of them is a "follower" forced to lose some of its observed output. In this choice of theirs, two main shortcomings can be identified. *First*, the ZSG-DEA efficiency scores cannot be used for ranking DMUs since they are computed relative to different efficient frontiers and thus, they are not really comparable across DMUs; s*econd*, their use does not guarantee that output interdependency is taken into consideration. To examine the former, consider Figure 1 where DMUs a and b, by being DEA efficient, require no output gain and thus, they are also ZSG-DEA efficient.[4] Then, if either DMU a or DMU b is under evaluation, the ZSG-DEA frontier actually coincides with the conventional DEA frontier. On the other hand, DMUs k and p require additional output (ck and dp, respectively) to become DEA efficient but, according to the output interdependency constraint, their output gains should be equal to the output losses of their competing DMUs, i.e., $c'k = aa' + bb' + pp'$ and $d''p = aa'' + bb'' + kk''$, respectively. Since by construction $aa' > aa''$ and $bb' > bb''$, it follows that $cc' > dd''$ and thus, the ZSG-DEA frontier ($T_{ZSG\text{-}DEA\_1}$) is farther than the conventional DEA frontier (T_{DEA}), if

DMU k is under evaluation, rather than it is ($T_{ZSG\text{-}DEA\_2}$), if DMU p is under evaluation. On the other hand, to examine the latter, compute the sum of DMUs' potential outputs as:

$$\sum_{i=1}^{I} \tilde{y}_i^i = \sum_{i=1}^{I} h^i y^i = \sum_{i=1}^{I} y^i + \sum_{i=1}^{I} z^i \tag{7}$$

From this sum, it then follows that the total potential output of DMUs equals their total observed output, i.e., $\sum_{i=1}^{I} \tilde{y}_i^i = \sum_{i=1}^{I} h^i y^i = \sum_{i=1}^{I} y^i$, only if *none* of the evaluated DMUs requires additional output to reach the ZSG-DEA frontier, i.e., $z^i = 0$ given that $h^i = 1 \ \forall\, i = 1, \dots, k, \dots, I$ or in other words, only if *all* DMUs are DEA efficient. In all other cases, output interdependency is not accounted for. Note in particular that, if $z^i > 0$ even for a single DMU, then $\sum_{i=1}^{I} \tilde{y}_i^i = \sum_{i=1}^{I} h^i y^i > \sum_{i=1}^{I} y^i$ implying that the evaluated DMUs appear as worse performers than they actually are.

3. An Alternative Ranking of DMUs for the ZSG-DEA Model

In contrast to the DMUs' ranking based solely on their ZSG-DEA efficiency scores, we propose an alternative ranking of DMUs' performance that is based on both their potential outputs, computed under the assumption that each DMU is the "leader", and their resultant outputs, computed under the assumption that each DMU is a "follower", from all "runs" of the ZSG-DEA model. For this purpose, we form the following $I \times I$ matrix of potential (diagonal elements) and resultant (off-diagonal elements) outputs:

$$\tilde{Y} = \begin{bmatrix} \tilde{y}_1^1 & \cdots & \tilde{y}_k^1 & \cdots & \tilde{y}_I^1 \\ \vdots & & \vdots & & \vdots \\ \tilde{y}_1^k & \cdots & \tilde{y}_k^k & \cdots & \tilde{y}_I^k \\ \vdots & & \vdots & & \vdots \\ \tilde{y}_1^I & \cdots & \tilde{y}_k^I & \cdots & \tilde{y}_I^I \end{bmatrix} \tag{8}$$

Note that each column of matrix $\tilde{Y}$ refers to a different "run" of (3) or (4) while each of its rows to a different DMU. That is, each column of matrix $\tilde{Y}$ represents the results, concerning all DMUs, that are obtained from the *same* "run" of (3) or (4) whereas each of its rows the results, concerning a given DMU, that are obtained from the I *different* "runs" of (3) or (4). More specifically, each column of matrix $\tilde{Y}$

indicates both the potential output of a particular "leader" and the $I-1$ resultant outputs of its "followers". On the other hand, each row of matrix $\tilde{Y}$ indicates both the potential output and the resultant outputs of a given DMU. Therefore, the superscript of $\tilde{y}$ corresponds to the DMU, to which this resultant output belongs, whereas its subscript to the DMU under evaluation or equivalently, to a specific "run" of (3) or (4). Given that (8) may be rewritten as:

$$\tilde{Y}=\begin{bmatrix} y^1+z^1 & \cdots & y^1-l_k^1 & \cdots & y^1-l_I^1 \\ \vdots & & \vdots & & \vdots \\ y^k-l_1^k & \cdots & y^k+z^k & \cdots & y^k-l_I^k \\ \vdots & & \vdots & & \vdots \\ y^I-l_1^I & \cdots & y^I-l_k^I & \cdots & y^I+z^I \end{bmatrix} \tag{9}$$

we can verify that the $\tilde{y}$'s in any column of matrix $\tilde{Y}$ are consistent with output interdependency by calculating their sum as:

$$\sum_{i=1}^{I}\tilde{y}_k^i=y^k+z^k+\sum_{i\neq k}^{I}\left(y^i-l_k^i\right) \tag{10}$$

This implies that the potential output of a given DMU $\left(y^k+z^k=\tilde{y}_k^k=h^k y^k\right)$ plus the sum of all other DMUs' resultant outputs $\left(\sum_{i\neq k}^{I}\left(y^i-l_k^i\right)=\sum_{i\neq k}^{I}\left(\tilde{y}_k^i\right)\right)$ are equal to the total (over DMUs) observed output due to the fact that $z^k=\sum_{i\neq k}^{I}l_k^i$. Thus, output interdependency is taken into consideration in each *separate* "run" of the ZSG-DEA model regardless of which output reduction strategy is employed.

To proceed with the development of the alternative metric of DMUs' performance, we first calculate for each evaluated DMU $(i=1,\dots,k,\dots,I)$ the simple arithmetic mean of its $\tilde{y}$'s in a given row of matrix $\tilde{Y}$ as:

$$\bar{y}^i=\frac{1}{I}\sum_{k=1}^{I}\tilde{y}_k^i=y^i+\frac{1}{I}\left(z^i-\sum_{k\neq i}^{I}l_k^i\right)=y^i+\bar{\eta}^i \tag{11}$$

which is equal to its observed output plus its average (over the I different "runs" of (3) or (4)) overall balance of output gain and output losses represented by $\bar{\eta}^i$. Note here that, if $z^i<\sum_{k\neq i}^{I}l_k^i$ implying that $\bar{\eta}^i<0$, then the i^{th} DMU may be considered as a relatively *high* performer since its additional output, which it gains from all other DMUs to reach the ZSG-DEA frontier, is smaller than the sum of its output losses caused by the output gains of its competing DMUs. On the other hand, if $z^i>$

$\sum_{k\neq i}^{I} l_k^i$ implying that $\bar{\eta}^i > 0$, then the i^{th} DMU may be considered as a relatively *low* performer since its required output gain is greater than the sum of its output losses. Lastly, if $z^i = \sum_{k\neq i}^{I} l_k^i$ implying that $\bar{\eta}^i = 0$, then the i^{th} DMU may be considered as a relatively *average* performer since its required output gain is equal to the sum of its output losses. Given that ZSG-DEA efficient DMUs require no output gain to reach the estimated frontier, they are necessarily high performers. On the other hand, depending on the sign of their $\bar{\eta}$'s, ZSG-DEA inefficient DMUs can be considered as relatively high, average or low performers. Then, by dividing both sides of (11) by y^i, we obtain the following performance metric for the i^{th} DMU:

$$\hat{h}^i = \frac{\bar{y}^i}{y^i} = 1 + \frac{\bar{\eta}^i}{y^i} \tag{12}$$

which is equal to the ratio of the average (over the I different ZSG-DEA frontiers) resultant output to the observed output of this DMU or equivalently, to one plus the relative average overall balance of output gain and output losses of this DMU. Thus, by considering all the estimated output gains, output losses and resultant outputs, our proposed performance metric is comparable across DMUs and for this reason, it can be readily used for ranking them. From (12), it follows that, if $\hat{h}^i = 1$, the i^{th} DMU is an *average* performer whereas, if $\hat{h}^i < (>)\ 1$, the i^{th} DMU is a *high* (*low*) performer. Importantly, as $\hat{h}^i$ becomes smaller (greater) than one, it indicates better (worse) performance for the i^{th} DMU.[5]

We can now express (11) and (12) in terms of either the proportional or the equal output reduction strategy. In the former case, we first write (9) as:

$$\tilde{Y} = \begin{bmatrix} h^1y^1 & \cdots & y^1RC_k & \cdots & y^1RC_I \\ \vdots & & \vdots & & \vdots \\ y^kRC_1 & \cdots & h^ky^k & \cdots & y^kRC_I \\ \vdots & & \vdots & & \vdots \\ y^IRC_1 & \cdots & y^IRC_k & \cdots & h^Iy^I \end{bmatrix} \tag{13}$$

and then we have:

$$\bar{y}^i = \frac{y^i\left(h^i + \sum_{k\neq i}^{I} RC_k\right)}{I} \tag{14}$$

and

$$\hat{h}^i = \frac{1}{I}\left(h^i + \sum_{k\neq i}^{I} RC_k\right) \tag{15}$$

Thus, in the case of the proportional output reduction strategy, our proposed performance metric for the i^{th} DMU is equal to the average of the ZSG-DEA efficiency score, estimated when this DMU is under evaluation, and the reduction coefficients, estimated when each other DMU is under evaluation.

On the other hand, in the latter case, we first write (9) as:

$$\tilde{Y} = \begin{bmatrix} h^1 y^1 & \cdots & y^1 - o_k & \cdots & y^1 - o_I \\ \vdots & & \vdots & & \vdots \\ y^k - o_1 & \cdots & h^k y^k & \cdots & y^k - o_I \\ \vdots & & \vdots & & \vdots \\ y^I - o_1 & \cdots & y^I - o_k & \cdots & h^I y^I \end{bmatrix} \tag{16}$$

and then we have:

$$\bar{y}^i = \frac{1}{I}\left[y^i\left(I - 1 + h^i\right) - \sum_{k\neq i}^{I} o_k\right] \tag{17}$$

and

$$\hat{h}^i = 1 + \frac{1}{I}\left[\left(h^i - 1\right) - \frac{\sum_{k\neq i}^{I} o_k}{y^i}\right] \tag{18}$$

Thus, in the case of the equal output reduction strategy, our proposed performance metric for the i^{th} DMU is equal to one plus the average difference between the ZSG-DEA efficiency score of this DMU minus one and the relative cumulative output loss of all other DMUs. Alternatively, (17) may be written as:

$$\bar{y}^i = \frac{1}{I}\left(I y^i + z^i - \frac{\sum_{k\neq i}^{I} z^k}{I-1}\right) = y^i + \frac{1}{I-1}\left(z^i - \bar{z}\right) \tag{19}$$

and in this case:

$$\hat{h}^i = 1 + \frac{z^i - \bar{z}}{y^i (I-1)} \tag{20}$$

From (20), it follows that the i^{th} DMU is a high (low) performer, if its required output gain is smaller (greater) than the average (over all DMUs) output gain $(\bar{z})$, while it is an average performer, if its required output gain is equal to $\bar{z}$. Therefore, a DMU that needs a smaller-than-the-average output gain to achieve its potentially best performance is in a better position compared to a DMU that needs the average output gain for the same purpose. In turn, the latter DMU is in a better position compared to a DMU that needs a greater-than-the-average output gain to achieve its potentially best performance.

Lastly, to verify that the $\hat{h}$'s in both (15) and (18) or (20) are consistent with output interdependency, we calculate the sum of DMUs' average resultant outputs as:

$$\sum_{i=1}^{I} \bar{y}^i = \begin{cases} \sum_{i=1}^{I} \left(\frac{1}{I} \sum_{k=1}^{I} \tilde{y}_k^i \right) = \frac{1}{I} \sum_{k=1}^{I} \sum_{i=1}^{I} \tilde{y}_k^i \\ \\ \sum_{i=1}^{I} \left(y^i + \frac{1}{I} \left(z^i - \sum_{k \neq i}^{I} l_k^i \right) \right) = \sum_{i=1}^{I} y^i + \frac{1}{I} \left(\sum_{i=1}^{I} z^i - \sum_{i=1}^{I} \sum_{k \neq i}^{I} l_k^i \right) \end{cases} \tag{21}$$

From (21), it follows that the total average resultant output of DMUs equals their total observed output ($\sum_{i=1}^{I} \bar{y}^i = \sum_{i=1}^{I} y^i$) because, as we have already shown in (10), $\sum_{i=1}^{I} \tilde{y}_k^i = \sum_{i=1}^{I} y^i$ and/or because the sum of DMUs' output gains is necessarily equal to the sum of their output losses, i.e., $\sum_{i=1}^{I} z^i = \sum_{i=1}^{I} \sum_{k \neq i}^{I} l_k^i$, due to the output interdependency constraint, which then is indeed reflected in our proposed performance metrics.

4. A Numerical Example

In this section, we illustrate the empirical applicability of our proposed performance metric by using a small dataset, presented in Table 2, of six hypothetical DMUs that employ two inputs to produce a single fixed-sum output. For our purposes, we first compute the ZSG-DEA efficiency scores of DMUs under the proportional and the equal output reduction strategy and then, we rank DMUs according to them. Specifically, from the second column of the upper part of Tables 3 and 4, it follows that DMUs C, D and F, which are ZSG-DEA efficient, are ranked first while DMUs E, B and A, which are ZSG-DEA inefficient, are ranked in the fourth, fifth and sixth position, respectively. Importantly, as it is evident from the ninth column of the upper part of Tables 3 and 4, this ranking is identical to the one based on DMUs' efficiency scores produced by the output-oriented VRS conventional DEA model.

Based on the reported ZSG-DEA efficiency scores that, being mostly estimated relative to different efficient frontiers, are not really comparable across DMUs, we also compute, under both the proportional and the equal reduction strategy, DMUs' potential outputs, presented in the third column of the upper part of Tables 3 and 4, respectively, whose sum (3962 and 4052, respectively) is higher than the total (over DMUs) observed output (3100). This means that output interdependency is not accounted for and thus, evaluated DMUs appear as worse performers than they actually are.

Then, by using the potential outputs of DMUs, we calculate their output gains and output losses under both the proportional and the equal reduction strategy. Specifically, from the fourth column of the upper part of Tables 3 and 4, it follows that DMUs C, D and F require no output gain while each of DMUs E, B and A gains additional output from all other DMUs to become ZSG-DEA efficient. Under the proportional reduction strategy, for example, DMU A gains 330 units of additional output and, in particular, 47 units from DMU B, 59 units from DMU C, 12 units from DMU D, 94 units from DMU E and 118 units from DMU F, as it is evident from the second column of the middle part of Table 3. On the other hand, under the equal reduction strategy, DMU A gains 345 units of additional output, i.e., 69 units from each of the other five DMUs, as it follows from the second column of the middle part of Table 4. Notice also that, under the proportional reduction strategy, DMU F, whose observed output is the highest among the evaluated DMUs, suffers the greatest output losses, whose sum is equal to 324 units, as it is evident from the eighth column of the middle part of Table 3. This DMU, in particular, loses 118 units of output from DMU A, 146 units from DMU B and 61 units from DMU E, as it follows from the seventh row of the middle part of Table 3. On the other hand, under the equal reduction strategy, not only DMU F but also the other ZSG-DEA efficient DMUs C and D suffer the greatest (among the evaluated DMUs) output losses, whose sums are equal to 190 units, as it is evident from the eighth column of the middle part of Table 4. Specifically, each of these DMUs loses 69 units of output from DMU A, 88 units from DMU B and 33 units from DMU E, as it follows from respectively the fourth, fifth and seventh row of the middle part of Table 4.

By combining some of the above information, we may estimate, under both the proportional and the equal reduction strategy, the overall balance of output gain and output losses of each evaluated DMU presented in the fifth column of the upper part of Tables 3 and 4, respectively. In addition, we can form the corresponding $\tilde{Y}$ matrix given in the lower part of Tables 3 and 4, respectively. It should be noted that, since $max(o_k) = 88 < min(y^k) = 100$, the equal reduction strategy causes no DMU to end up with a negative resultant output and thus, it can be readily used in this numerical example. Then, under both the proportional and the equal reduction strategy, we may calculate for each evaluated DMU the simple arithmetic mean of its $\tilde{y}$'s in a given row of the corresponding $\tilde{Y}$ matrix. In this way, we can compute the average (over all different ZSG-DEA frontiers) resultant outputs of DMUs, presented in the seventh column of the upper part of Tables 3 and 4, respectively, whose sum (3100) equals the total (over DMUs) observed output. According to (14), the average resultant outputs of DMUs can be also computed, under the proportional reduction strategy, by means of the estimated reduction coefficients given in the sixth column of the upper part of Table 3. On the other hand, under the equal reduction strategy, we may apply (17) or (19) to obtain the average resultant outputs of DMUs without using the corresponding matrix.

By dividing the average resultant output of each DMU by its observed output, we determine, under both the proportional and the equal reduction strategy, DMUs' alternative performance metrics, presented in the eighth column of the upper part of Tables 3 and 4, respectively, that, by taking account of all the different "runs" of the ZSG-DEA model, are both consistent with output interdependency and comparable across DMUs.[6] From these performance metrics, according to which we rank evaluated DMUs, it follows that there are 2 low performers, i.e., DMUs A and B, under all the different "runs" of (3) and 3 low performers, i.e., DMUs B, A and E, under all the different "runs" of (4), whose overall balances of output gain and output losses are positive. As a consequence, there are also 4 high performers, i.e., DMUs C, D, F and E, under all the different "runs" of (3) and 3 high performers, i.e., DMUs D, C and F, under all the different "runs" of (4), whose overall balances of output gain and output losses are negative. It is worth noting here that DMUs' ranking based on their alternative performance metrics provided under the proportional output reduction strategy is identical to their ranking based on either their corresponding ZSG-DEA efficiency scores or their conventional DEA efficiency scores. Therefore, none of the above sets of performance measures can completely differentiate evaluated DMUs. On the other hand, evaluated DMUs are fully discriminated only by their alternative performance metrics provided under the equal output reduction strategy. As a result, DMUs' ranking based on the specific measures is much more informative compared to all other reported rankings.

5. An Empirical Application: Sydney 2000 Olympic Games

To further illustrate the empirical applicability of our proposed performance metric, we use data from Sydney 2000 Olympic Games taken from Lins *et al.* (2003) appendix, which reports the number of gold, silver, and bronze medals of each of the 80 countries that won at least one of these medals. Based on these numbers, we compute countries' medal index, presented in the second column of Table 5, which is used in Lins *et al.* (2003) as the single fixed-sum output. This medal index is computed by means of a weighting scheme that, according to Lins *et al.* (2003), employs three different values, i.e., 0.5814, 0.2437 and 0.1749, as weights respectively assigned to the number of gold, silver, and bronze medals. From this medal index, whose mean is 3.8 implying that the average country won between 6.5 and 21.5 (gold, silver or bronze) medals, we can see that the most effective country in winning medals is U.S.A. while there are 11 countries, i.e., F.Y.R.O.M., Barbados, Canada, Qatar, Kyrgyzstan, Morocco, Armenia, Saudi Arabia, Vietnam, Chile and India, that won one bronze medal, 3 countries, i.e., Mexico, Nigeria and Sri Lanka, that won one silver medal, and 3 countries, i.e., Cameroon, Belgium and Portugal, that won one gold medal. Before proceeding, we should also note that Lins *et al.* (2003) consider the population and the gross domestic product of countries as their inputs to win medals but the values of these inputs are not reported in their paper.

In addition to the numbers of each country's medals, Lins *et al.* (2003) appendix reports the ranking of countries, also presented in the third column of Table 5, which is based on their ZSG-DEA efficiency scores produced by (3). By using these performance measures that, as we have already seen, are not really comparable across DMUs as they are mostly estimated relative to different ZSG-DEA frontiers, we compute countries' potential outputs, given in the fourth column of Table 5, whose sum (996.8) is higher than three times the total (over countries) observed output (305.2). This means that the ranking of countries reported in Lins *et al.* (2003) does not account for the output interdependency and thus, the evaluated countries appear as worse performers than they actually are.

Based on the potential outputs of countries, we also compute both their output gains, presented in the fifth column of Table 5, and their output losses under the proportional reduction strategy.[7] These values, in turn, are used for the estimation of each country's overall balance of output gain and output losses given in the sixth column of Table 5. In addition, they are used in the formation of the 80×80 $\tilde{Y}$ matrix, which is not reported here for brevity. By calculating for each country the simple arithmetic mean of its $\tilde{y}$'s in a given row of this matrix, we determine the average (over all different ZSG-DEA frontiers) resultant outputs of countries, presented in the eighth column of Table 5, whose sum (305.2) equals the total (over countries) observed output. According to (14), the average resultant outputs of countries can be determined also by means of the estimated reduction coefficients given in the seventh column of Table 5.

By dividing the average resultant output of each country by its observed output, we compute countries' alternative performance metrics, presented in the ninth column of Table 5, that, by taking account of all different ZSG-DEA frontiers, are both consistent with output interdependency and comparable across DMUs.[8] From these performance metrics, according to which we rank evaluated countries, it follows that 60% of them or, in other words, 48 out of the 80 countries have a positive overall balance of output gain and output losses and thus they are low performers, approximately 40% or, in other words, 31 out of the 80 countries have a negative overall balance of output gain and output losses and thus they are high performers while there is 1 country, i.e., Moldova, that is probably an average performer as it has an overall balance of output gain and output losses, which tends to zero. As it is further shown in Table 5, in the tenth column of which we find countries' conventional DEA efficiency scores estimated by means of (5b), there are 7 ZSG-DEA and conventional DEA efficient countries, i.e., Australia, Cuba, Russia, Bahamas, F.Y.R.O.M., U.S.A. and Barbados, with the rest of them (73) being distant from the estimated frontiers. Thus, high performers are significantly more than just the 7 ZSG-DEA efficient countries while low performers are much fewer than the 73 ZSG-DEA inefficient countries.

To emphasize that the evaluated countries are better discriminated by their alternative performance metrics than they are by both their conventional DEA efficiency scores and their ZSG-DEA efficiency scores, we are based on Figure 2, where the frequency distributions of all the aforementioned performance measures are graphically presented. From this figure, it follows that, except for the 31 high-performing countries whose alternative performance metrics are smaller than one, there are 27 countries with an alternative performance metric that lies between 1 and 1.09. On the other hand, there are only 7 countries with a conventional DEA efficiency score and a ZSG-DEA efficiency score that lie in this range of values. Moreover, there are 11 countries with an alternative performance metric that lies between 1.1 and 1.19 while there are no countries with either a conventional DEA efficiency score or a ZSG-DEA efficiency score that lies in this range of values. Lastly, there are only 11 countries with an alternative performance metric that is greater than or equal to 1.2 whereas both the conventional DEA efficiency score and the ZSG-DEA efficiency score of the vast majority of countries, i.e., 73 out of the 80, are not smaller than 1.2.

Before concluding this section, it is important to comment on the differences among the different rankings of countries that are also presented in Table 5. Specifically, in the first column of this table, we may find countries' medal ranking provided by the International Olympic Committee (IOC). This ranking sorts countries *first*, by the number of their gold medals, *second*, by the number of their silver medals, and *third*, by the number of their bronze medals. Besides, in the third, ninth and tenth column of Table 5, we can see the ranking of countries that is based respectively on their ZSG-DEA efficiency scores, their alternative performance metrics, and their conventional DEA efficiency scores. By comparing, for example, the ranking of countries that is based on their ZSG-DEA efficiency scores with their ranking based on their conventional DEA efficiency scores, we discover that in the former ranking Bulgaria, Poland, Czech Republic, Turkey, Uruguay and Portugal are ranked 1 position, Iceland, Iran and South Africa 2 positions, Belarus 3 positions, and Ireland 6 positions higher than they are ranked in the latter ranking. On the contrary, according to the ZSG-DEA efficiency scores, Yugoslavia, Sweden, Switzerland, Trinidad and Tobago, Belgium, Uzbekistan, Costa Rica, Mexico, Japan, Taiwan, Armenia, Argentina, Thailand and Colombia are ranked 1 position while Lithuania, Canada and Indonesia 2 positions lower than they are ranked according to the conventional DEA efficiency scores. Thus, it can be argued that the ranking based on the ZSG-DEA efficiency scores is more accurate than the ranking based on the conventional DEA efficiency scores. The reason is that, if they are considered *separately*, each of the former performance measures is consistent with output interdependency whereas the same cannot be said for each of the latter performance measures.

However, as we can see from our proposed performance metrics, not all of the above changes in the rank positions of countries are necessary given that the ZSG-DEA efficiency scores favor: Czech Republic as this country shouldn't be ranked higher than suggested by the conventional DEA efficiency scores; South Africa as this country should be actually ranked 1 position higher, instead of 2, than suggested by the conventional DEA efficiency scores; Belarus as this country should be actually ranked 2, instead of 3, positions higher than suggested by the conventional DEA efficiency scores; Ireland as this country should be actually ranked 5, instead of 6, positions higher than suggested by the conventional DEA efficiency scores. On the other hand, Sweden and Trinidad and Tobago are disadvantaged by the ZSG-DEA efficiency scores as they shouldn't be ranked lower than suggested by the conventional DEA efficiency scores. Similarly, Indonesia is also disadvantaged by the ZSG-DEA efficiency scores as it should be actually ranked 1 position lower, instead of 2, than suggested by the conventional DEA efficiency scores. Thus, it seems that the ranking based on our proposed performance metrics is more accurate than the ranking based on the ZSG-DEA efficiency scores since only the former performance measures, if considered *together*, are consistent with output interdependency.

Nevertheless, as Table 6 shows, the rankings of countries that are based on their ZSG-DEA efficiency scores and their alternative performance metrics are almost identical to each other. Moreover, they are very similar to the ranking of countries that is based on their conventional DEA efficiency scores whereas they are very different from countries' medal ranking, provided by the IOC, since this concentrates only on the achievements of countries without taking account of the different levels of their resources, such as wealth, population, etc., that are available for sports. These results follow from the estimated values of the statistic proposed by Saisana *et al.* (2005), i.e., $\bar{R}_S = \frac{1}{I}\sum_{i=1}^{I}\left|rank_{REF}^{i} - rank^{i}\right| \geq 0$, that is quantified as the average of the absolute differences in the rank positions of all DMUs with respect to a reference ranking. In particular, if there is no difference between any two rankings, $\bar{R}_S$ is equal to zero. On the other hand, the higher the value of this statistic, the greater is the aforementioned difference.

6. Discussion

In the previous sections of this paper, we took a closer look at the derivation and interpretation of the ZSG-DEA efficiency scores. Specifically, we explained that, in the estimation of these performance measures, only the different output gains of DMUs are taken into account and for this reason, the ZSG-DEA efficiency scores might be characterized as normative. Furthermore, we clarified that, even though output interdependency is taken into consideration in each separate "run" of the ZSG-DEA model, the total potential output of DMUs estimated by this model equals their total observed output, only if none of the evaluated DMUs requires additional output to reach the efficient frontier, and that, in any other case, output interdependency is actually not accounted for. Moreover, we underlined that all ZSG-DEA inefficient DMUs, which require different levels of additional output to become efficient, are evaluated based on different frontiers and as a result, their ZSG-DEA efficiency scores are not really comparable across them.

Our main objective was to prove both theoretically and empirically that, by taking account of all the different frontiers constructed by the ZSG-DEA model, our proposed performance metrics are able to consider all the estimated output gains, output losses and resultant outputs and as a result, they are both consistent with output interdependency, as the sum of DMUs' output gains is necessarily equal to the sum of their output losses, and comparable across DMUs, which then can be correctly ranked. In addition, we demonstrated that, if the alternative performance metric of a given DMU is smaller (greater) than one, implying that its required output gain is smaller (greater) than both the sum of its output losses and the average (over all DMUs) output gain, then this DMU is a relatively high (low) performer. Besides, we also showed that a DMU is a relatively average performer, if its alternative performance metric is equal to one, implying that its required output gain is equal to both the sum of its output losses and the average output gain. Therefore, we are able to conclude that: (*i*) a DMU, which needs a smaller (greater)-than-the-average output gain to achieve its potentially best performance, is in a better (worse) position compared to a DMU that needs the average output gain for the same purpose; (*ii*) ZSG-DEA efficient DMUs are necessarily high performers whereas ZSG-DEA inefficient DMUs can be characterized as relatively high, average or low performers; and (*iii*) as the alternative performance metric of a given DMU becomes smaller (greater) than one, it indicates better (worse) performance for this DMU.

According to our empirical results, presented in sections 4 and 5, evaluated DMUs are better discriminated by their alternative performance metrics than they are by both their conventional DEA efficiency scores and their ZSG-DEA efficiency scores. However, DMUs' ranking based on their alternative performance metrics, provided under the proportional output reduction strategy, is very similar to their ranking based on either their corresponding ZSG-DEA efficiency scores or their conventional DEA efficiency scores. Importantly, none of the above sets of performance measures can completely differentiate evaluated DMUs. On the other hand, evaluated DMUs are fully discriminated only by their alternative performance metrics provided under the equal output reduction strategy and as a result, DMUs' ranking based on the specific measures is much more informative compared to their ranking based on either their corresponding ZSG-DEA efficiency scores or their conventional DEA efficiency scores.

7. Concluding Remarks

In this paper, we stressed that it is better not to rank DMUs according to their ZSG-DEA efficiency scores since these are computed relative to different efficient frontiers and for this reason, they are not really comparable across DMUs. Moreover, we verified that DMUs' ranking, based on their ZSG-DEA efficiency scores, is also incompatible with output interdependency. For these reasons, we proposed an alternative metric of DMUs' performance that is easily computed, consistent with output interdependency, and comparable across DMUs. Based on its estimated values, we initially showed how to group DMUs as high, average or low performers and then, we ranked them within each group.

Using data from the Olympic Games, we compared and contrasted our alternative ranking for the ZSG-DEA model with the ranking based on the ZSG-DEA efficiency scores, provided under the proportional output reduction strategy, and we found that almost 1 out of 10 countries should be ranked differently than suggested by its ZSG-DEA efficiency score. It should be noted that the use of our proposed performance metric is limited for only ranking DMUs while this cannot be used for benchmarking purposes. That is, it cannot provide estimates of how much the output could be increased, if DMUs improved their performance.

Before concluding, it is important to also indicate some directions of future research on the ZSG-DEA model. For example, one research question to be answered in the future could be whether this model can employ a reverse fixed-sum output, whose larger (smaller) values reflect lower (higher) achievements for DMUs, and what happens with the zero-sum output redistribution strategies in this case. On the other hand, future researchers could also try to adapt the ZSG-DEA model to the case of a reverse fixed-sum input, whose larger (smaller) values reflect lower (higher) effort or expenditures for DMUs.

Declarations of interest: None

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Figure 1: The conventional DEA and the ZSG-DEA frontiers for the proportional output reduction strategy

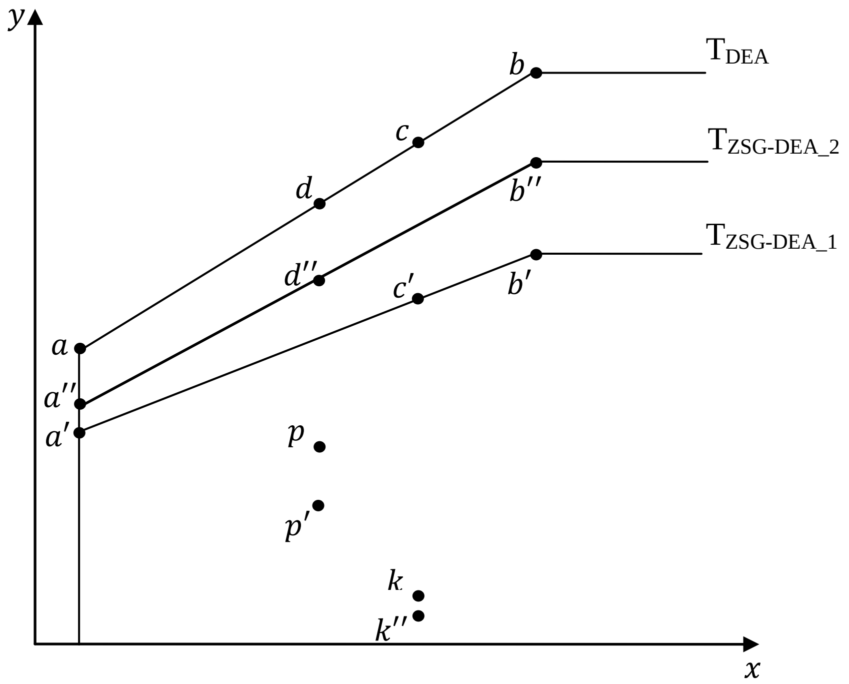

Figure 2: Frequency Distributions of the Different Performance Measures, Sydney 2000 Olympic Games

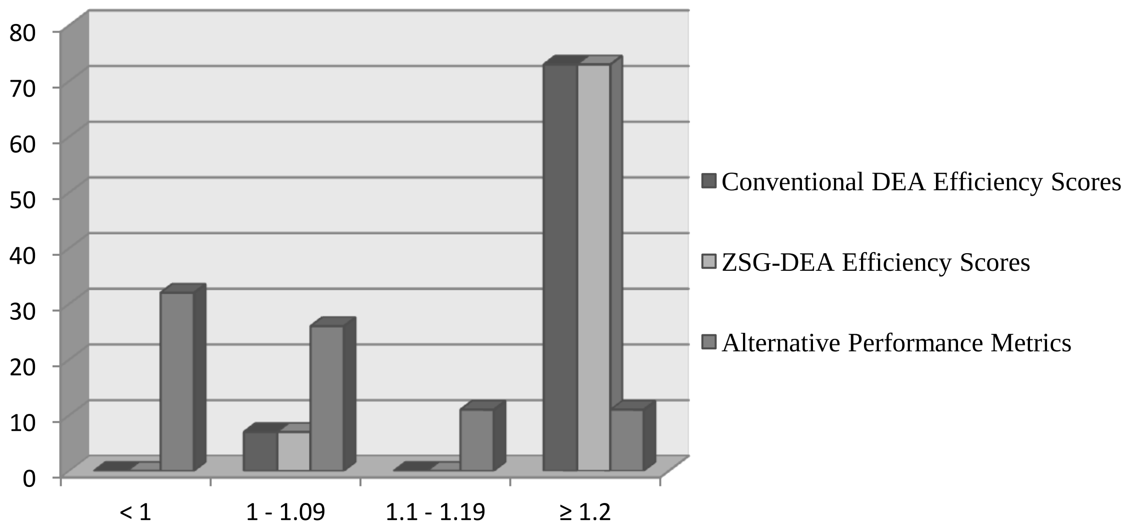

Table 1: Review of the Relevant Literature

Paper	Employed Methodology	Aim	Implication
Lins *et al.* (2003)	ZSG-DEA	Development of the ZSG-DEA model.	The ZSG-DEA model and the proportional/equal output reduction strategy are the first step to obtain more accurate results.
Gomes and Lins (2008)	ZSG-DEA	Evaluation based on the Uniform DEA frontier.	The ZSG-DEA models are especially suitable for treating equilibrium models.
Hu and Fang (2010)	ZSG-DEA	Efficiency measurement of DMUs when a fixed-sum output is used.	The conventional DEA model underestimates the efficiency scores of inefficient DMUs when a fixed-sum output is used.
Collier *et al.* (2011)	DEA	Correction to the problem of serial correlation among the conventional DEA efficiency scores when a fixed-sum output is used.	The conventional DEA efficiency scores are biased downward when a fixed-sum output is used.
Yang *et al.* (2011)	Fixed-Sum Output (FSO) DEA	Development of a DEA model to accommodate competition over outputs.	In the presence of competition over outputs, the best-practice frontier deviates from the conventional DEA frontier.
Bi *et al.* (2014)	ZSG-DEA	Transformation of the non-linear ZSG-DEA models to linear or parametric linear models.	The linear formulations of ZSG-DEA models under the equal reduction strategy and the proportional reduction strategy in a single output case are equivalent to the output-oriented super-efficiency model under Variable-Returns-to-Scale (VRS) assumption.
Yang *et al.* (2014)	EEF DEA	Development of the EEF DEA approach, by which all DMUs with fixed-sum outputs can be evaluated based on a common platform.	The proposed approach simultaneously satisfies some important conditions that previous methods do not satisfy when evaluating competitive DMUs with fixed-sum outputs.
Bi *et al.* (2015)	Output-Constrained (OC) DEA	Efficiency evaluation of a production system with an upper bounded sum of outputs.	The OC-DEA efficiency scores are not less than the Constant-Returns-to-Scale (CRS) conventional DEA efficiency scores.
Yang *et al.* (2015)	EEF DEA	Development of the Generalized EEF DEA approach.	Compared to the EEF DEA approach, the proposed approach makes several improvements in DMUs' evaluation.
Fang (2016)	EEF DEA	Development of a linear programming model that can achieve a common EEF in a single step.	The proposed approach can overcome one limitation of previous relevant approaches since it may provide a unique EEF.
Amirteimoori *et al.* (2017)	Context-Dependent EEF DEA	Construction of multiple EEFs.	A single EEF needs a significant trade-off between efficient and inefficient DMUs and this may be impossible in practical applications.
Zhu *et al.* (2017)	EEF DEA	Development of an algorithm based on the secondary goal approach to address the problem of non-uniqueness of the EEF.	The proposed algorithm is proven mathematically to be an effective approach to guaranteeing the uniqueness of the EEF.
Feng *et al.* (2019)	ZSG-DEA	Development of two improved ZSG-DEA models assuming CRS and VRS, respectively.	Based on the Uniform DEA frontier, the proposed method can always allocate multiple resources/quotas under both CRS and VRS.
Wu *et al.* (2019)	EEF DEA	Development of the common EEF DEA approach with fixed-sum undesirable output.	The proposed approach simultaneously satisfies some important conditions that previous methods do not satisfy when evaluating competitive DMUs with fixed-sum undesirable outputs.

Continue			
Bernardo *et al.* (2020)	Conventional DEA and ZSG-DEA	Development of a two-step approach that is based on the CRS conventional DEA model in the first stage and the CRS ZSG-DEA model in the second stage.	The proposed approach provides an instrument for improving resource utilization when a fixed-sum input is used.
Amirteimoori *et al.* (2020)	EEF DEA	Development of a DEA-based model to construct an EEF in the presence of multi-type input/output variables.	The proposed model can easily be used in all real cases, in which a set of homogeneous DMUs compete under the supervision of a central decision-maker.
Zhu *et al.* (2020a)	EEF DEA	Development of an extended secondary goal approach to address the problem of non-uniqueness of the EEF.	The proposed approach differs from the conventional secondary goal approaches since it can consider each DMU's minimum and maximum inefficiency value.
Zhu *et al.* (2020b)	EEF DEA	Construction of a new Generalized EEF that is based on minimum satisfaction degree maximization of all DMUs.	The new Generalized EEF considers both minimum and maximum adjustment strategy while previous relevant studies only minimized the weighted sum reduction.
Liu *et al.* (2021)	ZSG-DEA	Development of a new algorithm for constructing the Uniform DEA Frontier.	The proposed algorithm simplifies the construction of the Uniform DEA Frontier and extends from single to multiple resource allocation.
Mohamadinejad *et al.* (2021)	EEF DEA	Classification of DMUs into different classes based on their size and construction of a common EEF in each class.	The input/output tradeoff is more rational among DMUs of similar size.
Yang *et al.* (2021)	EEF DEA	Development of a new Generalized EEF DEA approach that considers fixed-sum undesirable outputs.	The proposed approach simultaneously satisfies some important conditions that previous methods do not satisfy when evaluating competitive DMUs with fixed-sum undesirable outputs
Zhu *et al.* (2021)	EEF DEA	Construction of a new Common EEF that considers each DMU's own adjustment strategy for the fixed-sum input/output.	The new Common EEF allows a more accurate evaluation of DMUs' efficiency when a fixed-sum input/output is used.
Chen *et al.* (2021)	EEF DEA	Performance evaluation based on all the feasible EEFs by means of several models that provide the corresponding efficiency intervals, ranking intervals, and dominance relations for the DMUs with fixed-sum outputs.	When evaluating competitive DMUs with fixed-sum outputs, the proposed approach gives more informative results than previous DEA approaches.
Li *et al.* (2021a)	EEF DEA and Two-Stage DEA	Development of a Two-Stage DEA model with fixed-sum final outputs.	The proposed model gives the overall efficiency and its decomposition for each DMU and achieves a complete efficiency ranking both for the overall production process and for its individual stages.
Li *et al.* (2021b)	EEF DEA and Malmquist DEA	Development of a Generalized EEF DEA model with fixed-sum undesirable outputs that is combined with the Malmquist productivity index.	Considering fixed-sum undesirable outputs is not a trivial concern because distortions occur in approaches omitting consideration of fixed-sum undesirable outputs.

Table 2: Data and Descriptive Statistics, Example of Six Hypothetical DMUs

DMU	y^k	x_1^k	x_2^k
A	300	1000	500
B	400	1000	800
C	500	500	200
D	100	220	1000
E	800	400	1000
F	1000	300	900
Sum	3100		
Average	516.667	570	733.333
Standard Deviation	331.160	346.121	320.416

Table 3: Proportional Output Reduction, Example of Six Hypothetical DMUs

DMU	h^k	$h^k y^k$	z^k	η^i	RC_k	$\bar{y}^i$	$\hat{h}^i$	θ^k
A	2.100 (6)	630.081	330.081	268.234	0.882	344.706	1.149 (6)	2.381 (6)
B	1.982 (5)	792.991	392.991	321.594	0.854	453.599	1.134 (5)	2.320 (5)
C	1.000 (1)	500	0.000	-162.022	1.000	472.996	0.946 (1)	1.000 (1)
D	1.000 (1)	100	0.000	-32.404	1.000	94.599	0.946 (1)	1.000 (1)
E	1.174 (4)	939.394	139.394	-71.357	0.939	788.107	0.985 (4)	1.250 (4)
F	1.000 (1)	1000	0.000	-324.044	1.000	945.993	0.946 (1)	1.000 (1)
Sum		3962.466	862.466	0.000		3100		
Average	1.376	660.411	143.744	0.000	0.946	516.667	1.018	1.492
Standard Deviation	0.521	332.097	178.242	250.108	0.065	307.115	0.097	0.672
l_k^i	A	B	C	D	E	F	$\sum_{k\neq i}^{I} l_k^i$	
A	0.000	43.666	0.000	0.000	18.182	0.000	61.847	
B	47.154	0.000	0.000	0.000	24.242	0.000	71.397	
C	58.943	72.776	0.000	0.000	30.303	0.000	162.022	
D	11.789	14.555	0.000	0.000	6.061	0.000	32.404	
E	94.309	116.442	0.000	0.000	0.000	0.000	210.751	
F	117.886	145.552	0.000	0.000	60.606	0.000	324.044	
$\tilde{y}_k^i$	A	B	C	D	E	F	$\sum_{k=1}^{I} \tilde{y}_k^i$	
A	630.081	256.334	300	300	281.818	300	2068.234	
B	352.846	792.991	400	400	375.758	400	2721.594	
C	441.057	427.224	500	500	469.697	500	2837.978	
D	88.211	85.445	100	100	93.939	100	567.596	
E	705.691	683.558	800	800	939.394	800	4728.643	
F	882.114	854.448	1000	1000	939.394	1000	5675.956	

Note: In the second, eighth and ninth column of the upper part, the numbers in the brackets refer to the positions of countries in the ranking based respectively on the ZSG-DEA efficiency scores, our proposed performance metrics, and the conventional DEA efficiency scores.

Table 4: Equal Output Reduction, Example of Six Hypothetical DMUs

DMU	h^k	$h^k y^k$	z^k	η^i	o_k	$\bar{y}^i$	$\hat{h}^i$	θ^k
A	2.151 (6)	645.238	345.238	223.892	69.048	337.315	1.124 (5)	2.381 (6)
B	2.100 (5)	840.062	440.062	337.681	88.012	456.280	1.141 (6)	2.320 (5)
C	1.000 (1)	500	0.000	-190.393	0.000	468.268	0.937 (2)	1.000 (1)
D	1.000 (1)	100	0.000	-190.393	0.000	68.268	0.683 (1)	1.000 (1)
E	1.208 (4)	966.667	166.667	9.607	33.333	801.601	1.002 (4)	1.250 (4)
F	1.000 (1)	1000	0.000	-190.393	0.000	968.268	0.968 (3)	1.000 (1)
Sum		4051.967	951.967	0.000	190.393	3100		
Average	1.410	675.328	158.661	0.000	31.732	516.667	0.976	1.492
Standard Deviation	0.560	340.610	194.721	233.665	38.944	323.836	0.166	0.672

o_k	A	B	C	D	E	F	$\sum_{k\neq i}^{I} l_k^i$
A	0.000	88.012	0.000	0.000	33.333	0.000	121.346
B	69.048	0.000	0.000	0.000	33.333	0.000	102.381
C	69.048	88.012	0.000	0.000	33.333	0.000	190.393
D	69.048	88.012	0.000	0.000	33.333	0.000	190.393
E	69.048	88.012	0.000	0.000	0.000	0.000	157.060
F	69.048	88.012	0.000	0.000	33.333	0.000	190.393

$\tilde{y}_k^i$	A	B	C	D	E	F	$\sum_{k=1}^{I} \tilde{y}_k^i$
A	645.238	211.988	300	300	266.667	300	2023.892
B	330.952	840.062	400	400	366.667	400	2737.681
C	430.952	411.988	500	500	466.667	500	2809.607
D	30.952	11.988	100	100	66.667	100	409.607
E	730.952	711.988	800	800	966.667	800	4809.607
F	930.952	911.988	1000	1000	966.667	1000	5809.607

Note: In the second, eighth and ninth column of the upper part, the numbers in the brackets refer to the positions of countries in the ranking based respectively on the ZSG-DEA efficiency scores, our proposed performance metrics, and the conventional DEA efficiency scores.

Table 5: Performance Rankings, Sydney 2000 Olympic Games

Proportional Output Reduction Strategy									
Country	y^k	h^k	$h^k y^k$	z^k	η^i	RC_k	$\bar{y}^i$	$\hat{h}^i$	θ^k
Australia (4)	18.36820	1.00000 (1)	18.36820	0.00000	-42.07467	1.00000	17.84227	0.97137 (1)	1.00000 (1)
Cuba (9)	10.30040	1.00000 (1)	10.30040	0.00000	-23.59436	1.00000	10.00547	0.97137 (1)	1.00000 (1)
Russia (2)	30.32560	1.00000 (1)	30.32560	0.00000	-69.46460	1.00000	29.45729	0.97137 (1)	1.00000 (1)
Bahamas (34)	0.82510	1.00000 (1)	0.82510	0.00000	-1.89000	1.00000	0.80148	0.97137 (1)	1.00000 (1)
F.Y.R.O.M. (71)	0.17490	1.00000 (1)	0.17490	0.00000	-0.40063	1.00000	0.16989	0.97137 (1)	1.00000 (1)
U.S.A. (1)	34.53880	1.00000 (1)	34.53880	0.00000	-79.11547	1.00000	33.54986	0.97137 (1)	1.00000 (1)
Barbados (71)	0.17490	1.00000 (1)	0.17490	0.00000	-0.40063	1.00000	0.16989	0.97137 (1)	1.00000 (1)
Romania (11)	1.49950	1.31234 (8)	1.96785	0.46835	-2.96413	0.99846	1.46245	0.97529 (8)	1.31436 (8)
Norway (19)	9.43170	1.34066 (9)	12.64472	3.21302	-18.28902	0.98914	9.20309	0.97576 (9)	1.35538 (9)
China (3)	3.58140	1.36893 (10)	4.90267	1.32127	-6.86669	0.99562	3.49557	0.97603 (10)	1.37495 (10)
Germany (5)	22.80190	1.40746 (11)	32.09275	9.29085	-42.18968	0.96710	22.27453	0.97687 (11)	1.45533 (11)
Hungary (13)	16.82990	1.45858 (12)	24.54770	7.71780	-30.38284	0.97324	16.45011	0.97743 (12)	1.49868 (12)
Netherlands (8)	6.63810	1.56421 (13)	10.38339	3.74529	-11.37685	0.98746	6.49589	0.97858 (13)	1.58408 (13)
France (6)	9.86970	1.66279 (14)	16.41121	6.54151	-15.84770	0.97785	9.67160	0.97993 (14)	1.70045 (14)
Bulgaria (16)	12.89390	1.67926 (15)	21.65223	8.75833	-20.39048	0.97004	12.63902	0.98023 (15)	1.73112 (16)
Yugoslavia (44)	4.71900	1.69924 (16)	8.01869	3.29969	-7.45796	0.98902	4.62578	0.98024 (16)	1.71810 (15)
Italy (7)	1.00000	1.80538 (17)	1.80538	0.80538	-1.48260	0.99735	0.98147	0.98147 (17)	1.81017 (17)
Estonia (47)	11.78150	1.91939 (18)	22.61324	10.83174	-15.72039	0.96309	11.58500	0.98332 (18)	1.99295 (18)
Slovenia (36)	0.93120	2.05719 (19)	1.91566	0.98446	-1.14556	0.99676	0.91688	0.98462 (19)	2.06387 (19)
U.K. (10)	1.16280	2.11238 (20)	2.45627	1.29347	-1.36512	0.99575	1.14574	0.98533 (20)	2.12140 (20)
Belarus (23)	10.05670	2.19780 (21)	22.10264	12.04594	-10.57980	0.95919	9.92445	0.98685 (22)	2.29131 (24)
Sweden (18)	4.39920	2.21631 (22)	9.75000	5.35080	-4.64788	0.98221	4.34110	0.98679 (21)	2.25645 (21)
Jamaica (54)	4.06880	2.23964 (23)	9.11265	5.04385	-4.20810	0.98325	4.01620	0.98707 (23)	2.27779 (23)
Lithuania (33)	1.68750	2.25683 (24)	3.80840	2.12090	-1.73275	0.99301	1.66584	0.98716 (24)	2.27271 (22)
South Korea (12)	8.76840	2.29253 (25)	20.10179	11.33339	-8.41654	0.96177	8.66319	0.98800 (25)	2.38365 (25)
Greece (17)	4.31250	2.37586 (26)	10.24590	5.93340	-3.85989	0.98028	4.26425	0.98881 (26)	2.42365 (26)
Ukraine (21)	5.93020	2.52334 (27)	14.96392	9.03372	-4.37117	0.96982	5.87556	0.99079 (27)	2.60187 (27)
Denmark (30)	2.06880	2.72183 (28)	5.63092	3.56212	-1.15242	0.98825	2.05439	0.99304 (28)	2.75419 (28)
Latvia (44)	1.00000	2.76778 (29)	2.76778	1.76778	-0.51703	0.99419	0.99354	0.99354 (29)	2.78396 (29)
Poland (14)	5.23160	3.08833 (30)	16.15689	10.92529	-0.86783	0.96358	5.22075	0.99793 (30)	3.20504 (31)
Switzerland (37)	2.39340	3.14070 (31)	7.51696	5.12356	-0.31833	0.98308	2.38942	0.99834 (31)	3.19475 (30)
Moldova (61)	0.41860	3.27761 (32)	1.37201	0.95341	-0.00414	0.99687	0.41855	0.99988 (32)	3.28790 (32)
Ethiopia (20)	3.09400	3.32005 (33)	10.27224	7.17824	0.16455	0.97624	3.09606	1.00066 (33)	3.40085 (33)

Continue

Finland (31)	1.58140	3.43407 (34)	5.43063	3.84923	0.24688	0.98732	1.58449	1.00195 (34)	3.47816 (34)
Slovakia (39)	1.48740	3.62188 (35)	5.38718	3.89978	0.51180	0.98716	1.49380	1.00430 (35)	3.66898 (35)
N. Zealand (46)	1.10610	3.72301 (36)	4.11802	3.01192	0.48921	0.99010	1.11222	1.00553 (36)	3.76025 (36)
Czech Rep. (28)	2.41860	4.05680 (37)	9.81176	7.39316	1.91211	0.97559	2.44250	1.00988 (38)	4.15832 (38)
Trinidad and Tobago (61)	0.41860	4.07332 (38)	1.70509	1.28649	0.32940	0.99578	0.42272	1.00984 (37)	4.09058 (37)
Kazakhstan (22)	2.71900	4.20875 (39)	11.44360	8.72460	2.57481	0.97116	2.75119	1.01184 (39)	4.33374 (39)
Kenya (29)	2.24370	4.51264 (40)	10.12500	7.88130	2.80019	0.97399	2.27870	1.01560 (40)	4.63315 (40)
Iceland (71)	3.87450	4.70810 (41)	18.24153	14.36703	5.67671	0.95233	3.94546	1.01831 (41)	4.94379 (43)
Georgia (68)	1.04940	4.76190 (42)	4.99714	3.94774	1.55758	0.98702	1.06887	1.01855 (42)	4.82452 (42)
Canada (24)	0.17490	4.78240 (43)	0.83644	0.66154	0.26129	0.99783	0.17817	1.01867 (43)	4.79279 (41)
Azerbaijan (34)	1.33770	5.33903 (44)	7.14202	5.80432	2.76570	0.98090	1.37227	1.02584 (44)	5.44299 (44)
Spain (25)	1.40650	5.75043 (45)	8.08798	6.68148	3.49065	0.97801	1.45013	1.03102 (45)	5.87973 (45)
Austria (32)	0.75630	5.85823 (46)	4.43058	3.67428	1.95101	0.98793	0.78069	1.03225 (46)	5.92979 (46)
Croatia (49)	3.34980	5.96659 (47)	19.98687	16.63707	9.14854	0.94489	3.46416	1.03414 (47)	6.31459 (47)
Qatar (71)	0.17490	6.36943 (48)	1.11401	0.93911	0.53902	0.99692	0.18164	1.03852 (48)	6.38910 (48)
Kyrgyzstan (71)	0.17490	8.67303 (49)	1.51691	1.34201	0.94215	0.99560	0.18668	1.06734 (49)	8.71135 (49)
Cameroon (50)	0.58140	9.24214 (50)	5.37338	4.79198	3.46936	0.98427	0.62477	1.07459 (50)	9.38984 (50)
Algeria (42)	1.34980	9.57854 (51)	12.92912	11.57932	8.53887	0.96190	1.45654	1.07908 (51)	9.95799 (51)
Mozambique (50)	1.01210	9.82318 (52)	9.94204	8.92994	6.64131	0.97065	1.09512	1.08202 (52)	10.12025 (52)
Iran (27)	1.91910	9.89120 (53)	18.98220	17.06310	12.77512	0.94374	2.07879	1.08321 (53)	10.48080 (55)
Belgium (55)	0.58140	9.95025 (54)	5.78507	5.20367	3.88184	0.98292	0.62992	1.08346 (54)	10.12316 (53)
Uzbekistan (43)	1.17490	10.01001 (55)	11.76076	10.58586	7.93551	0.96518	1.27409	1.08443 (55)	10.37108 (54)
Turkey (26)	2.51160	10.53741 (56)	26.46575	23.95415	18.39976	0.92087	2.74160	1.09157 (56)	11.44287 (57)
Costa Rica (69)	0.34980	10.96491 (57)	3.83553	3.48573	2.68846	0.98857	0.38341	1.09607 (57)	11.09172 (56)
Brazil (53)	1.91910	11.14827 (58)	21.39465	19.47555	15.20283	0.93579	2.10914	1.09902 (58)	11.91320 (58)
Morocco (58)	0.17490	11.91895 (59)	2.08462	1.90972	1.51019	0.99374	0.19378	1.10793 (59)	11.99404 (59)
Kuwait (71)	0.94330	12.43781 (60)	11.73259	10.78929	8.66199	0.96454	1.05157	1.11478 (60)	12.89503 (60)
North Korea (60)	1.59350	13.81215 (61)	22.00967	20.41617	16.87320	0.93276	1.80441	1.13236 (61)	14.80780 (61)
Uruguay (64)	0.76840	14.47178 (62)	11.12012	10.35172	8.61772	0.96600	0.87612	1.14019 (62)	14.98113 (63)
Mexico (40)	0.24370	14.77105 (63)	3.59970	3.35600	2.80046	0.98900	0.27871	1.14364 (63)	14.93539 (62)
South Africa (55)	1.66230	16.58375 (64)	27.56716	25.90486	22.23901	0.91467	1.94029	1.16723 (65)	18.13092 (66)
Nigeria (41)	0.24370	16.58375 (64)	4.04146	3.79776	3.24257	0.98755	0.28423	1.16632 (64)	16.79285 (64)
Ireland (64)	5.73110	16.61130 (66)	95.20100	89.46990	78.05413	0.70127	6.70678	1.17024 (67)	23.68735 (72)
Indonesia (38)	0.73110	16.69449 (67)	12.20534	11.47424	9.82711	0.96232	0.85394	1.16802 (66)	17.34820 (65)
Japan (15)	1.01210	17.36111 (68)	17.57118	16.55908	14.29583	0.94557	1.19080	1.17656 (68)	18.36049 (67)
Taiwan (58)	0.83720	18.72659 (69)	15.67790	14.84070	12.96381	0.95125	0.99925	1.19356 (69)	19.68639 (68)
Armenia (71)	0.17490	21.09705 (70)	3.68987	3.51497	3.11636	0.98848	0.21385	1.22272 (70)	21.34297 (69)

Continue									
Argentina (57)	0.94330	21.14165 (71)	19.94292	18.99962	16.89777	0.93756	1.15452	1.22392 (71)	22.54962 (70)
Thailand (47)	0.93120	21.55172 (72)	20.06897	19.13777	17.06330	0.93711	1.14449	1.22905 (72)	22.99808 (71)
Portugal (69)	0.58140	27.39726 (73)	15.92877	15.34737	14.04489	0.94962	0.75696	1.30196 (73)	28.85065 (74)
Colombia (50)	0.34980	27.93296 (74)	9.77095	9.42115	8.63070	0.96910	0.45768	1.30842 (74)	28.82363 (73)
Saudi Arabia (61)	0.17490	34.12969 (75)	5.96928	5.79438	5.39707	0.98101	0.24236	1.38573 (75)	34.79051 (75)
Israel (71)	0.41860	34.84321 (76)	14.58537	14.16677	13.22736	0.95352	0.58394	1.39499 (76)	36.54152 (76)
Vietnam (64)	0.17490	43.66812 (77)	7.63755	7.46265	7.06630	0.97554	0.26323	1.50502 (77)	44.76316 (77)
Sri Lanka (64)	0.24370	57.14286 (78)	13.92571	13.68201	13.13472	0.95514	0.40788	1.67371 (78)	59.82670 (78)
Chile (71)	0.17490	76.92308 (79)	13.45385	13.27895	12.88593	0.95647	0.33597	1.92095 (79)	80.42384 (79)
India (71)	0.17490	161.29032 (80)	28.20968	28.03478	27.65022	0.90810	0.52053	2.97614 (80)	177.61280 (80)
Sum	305.23490		996.75869	691.52379	0.00000		305.23490		
Average	3.81544	11.63603	12.45948	8.64405	0.00000	0.97137	3.81544	1.10468	12.27801
Standard Deviation	6.37398	21.25729	12.59059	11.27517	19.06358	0.03750	6.18491	0.26586	23.05769

Note: In the first column, the numbers in the brackets refer to the positions of countries in the medal ranking provided by the International Olympic Committee (IOC). In the third, ninth and tenth column, the numbers in the brackets refer to the positions of countries in the ranking based respectively on the ZSG-DEA efficiency scores, our proposed performance metrics, and the conventional DEA efficiency scores.

Source: Lins *et al.* (2003) appendix; own calculations.

Table 6: Average Shifts in Rank Positions, Sydney 2000 Olympic Games

$\bar{R}_S$	$\hat{h}$	θ	IOC
h	0.09	0.51	13.89
$\hat{h}$		0.43	13.90
θ			13.95

Note: h refers to the ranking of countries that is based on their ZSG-DEA efficiency scores, $\hat{h}$ to their ranking based on our proposed performance metrics, θ to their ranking based on their conventional DEA efficiency scores, and IOC to their medal ranking provided by the International Olympic Committee.

Footnotes

[1] The ZSG-DEA model is solved *separately* for each evaluated DMU and thus, its "runs" are as many as the number of the DMUs under evaluation.

[2] Other interesting cases concerning the ZSG-DEA model are considered in Bi *et al.* (2014) and in Bouzidis and Karagiannis (2021a, 2021b).

[3] Gomes and Lins (2008) coined the name TAT since this is simply referred to as Theorem in Lins *et al.* (2003).

[4] Since, according to Lins *et al.* (2003), Bi *et al.* (2014) and Bouzidis and Karagiannis (2021b), (3), (4) and the corresponding conventional DEA model produce identical sets of intensity variables, the efficient frontiers obtained from these models are constructed by the same DMUs.

[5] Therefore, our proposed performance metrics resemble the super-efficiency DEA scores of Andersen and Petersen (1993), who however developed their proposed ratio-type performance measures in the input space and for this reason, in their paper, high performers or super-efficient DMUs end up with a super-efficiency DEA score that is greater than one while low performers or inefficient DMUs with a conventional DEA efficiency score that is smaller than one.

[6] Depending on the employed (proportional or equal) output reduction strategy, these performance metrics can be determined also by means of (15), (18) or (20).

[7] The equal reduction strategy is not used in Lins *et al*. (2003) as it causes countries with a low medal index, such as those that won only one bronze medal, to end up with a negative resultant output when the estimated output loss is big enough.

[8] These performance metrics can be computed also by means of (15).