Extending the Zero-Sum Gains Data Envelopment Analysis Model

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ABSTRACT: In this paper, we adapt the ZSG-DEA model to the case of a *reverse* output, whose larger (smaller) values reflect lower (higher) achievements. Then, we introduce the zero-sum reverse output redistribution strategies, state the resulting Target's Assessment Theorem for both the proportional and the equal *expansion* strategy, and confirm that the Benchmarks' Contribution Equality Theorem is also applicable to these cases. We also apply the ZSG-DEA model with a forward and a reverse output to estimate respectively teams' offensive and defensive efficiency in Greek premier soccer league.

KEYWORDS: Data Envelopment Analysis (DEA); Zero-Sum Gains (ZSG); Output Interdependency; Forward/Reverse Fixed-Sum Output; Offensive/Defensive Efficiency; Soccer Teams

JEL CLASSIFICATION CODES: Z2, C14, C61, L83

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1. Introduction

For accurately assessing the performance of Decision Making Units (DMUs) under output interdependency, i.e., when their total actual output is *a priori* fixed and thus the output of one DMU depends on the outputs of all others, one could employ the Zero-Sum Gains Data Envelopment Analysis (ZSG-DEA) model developed by Lins *et al.* (2003). Besides non-competitive market models, output interdependency is evident (*i*) if DMUs' market shares (Hu and Fang, 2010; Amirteimoori *et al.*, 2017) are considered as the relevant output variable, (*ii*) when analyzing the quota allocation in the case of a fixed-sum undesirable output, such as emission permits (Gomes and Lins, 2008; Feng *et al.*, 2019), and (*iii*) in the case of head-to-head competitions, such as in sports where teams' or athletes' achievements are necessarily related to those of their competitors (see Lins *et al.* (2003) for the Olympic Games and Collier *et al.* (2011) for American football) or in political elections (Sexton and Lewis, 2012). In these cases, output interdependency affects DMUs' performance in the sense that the resulting efficiency scores should imply the total (across DMUs) potential output to be fixed at the level of the total (across DMUs) actual output. If however the former is proved to be greater than the latter one, then output interdependency is not reflected in the estimated efficiency scores and consequently, most evaluated DMUs seem less efficient than they really are (Lins *et al.*, 2003).

The ZSG-DEA model guarantees that output interdependency is accounted for in each of its "runs" and irrespective of whether the DMU under evaluation is efficient or not.[1] This is achieved by ensuring that the additional output, which each evaluated DMU may need to become efficient, is gained from all other DMUs, whose output losses can be modeled by means of three alternative output reduction strategies, namely, the *proportional* (Lins *et al.*, 2003), the *equal* (Lins *et al.*, 2003; Collier *et al.*, 2011), and the *minimal* (Yang *et al.*, 2011). The proportional reduction strategy assumes that the output loss of each DMU is proportional to its actual output, the equal reduction strategy assumes equal output losses for all DMUs, and the minimal reduction strategy assumes minimal output losses for all DMUs. Regardless of the reduction strategy, the output gain and output losses of DMUs are obviously interdependent and for this reason, they become choice variables making non-linear the ZSG-DEA model. However, for several of its variants,[2] it is possible to obtain a linear formulation (see Bi *et al*. (2014)) or even to compute the ZSG-DEA efficiency scores directly from the corresponding conventional DEA efficiency scores if Lins *et al*. (2003) Target's Assessment Theorem (TAT) and Benchmarks' Contribution Equality Theorem (BCET) hold.[3]

Considering in particular the case of soccer teams, output interdependency matters *first* in evaluating athletic efficiency, i.e., the success of converting attacking moves into accumulated points during a league season, as the total (across all teams) number of points is given and thus, total potential accumulated points cannot exceed total actual accumulated points. *Second*, in assessing offensive and defensive efficiency as the total number of goals scored and the total number of goals conceded during a league season are necessarily equal to each other. This is because, if a team scores a goal and thus increases its offensive output, then another team should inevitably concede a goal at the same time and thus decrease its defensive output. Therefore, when assessing teams' offensive efficiency during a league season reflecting the relation between their actions in offense and goals scored, their total potential goals scored should not exceed their total actual goals conceded. Similarly, when assessing teams' defensive efficiency during a league season reflecting the relation between their actions in defense and goals conceded, their total potential goals conceded should not fall short of their total actual goals scored.

Even though the ZSG-DEA model in its present form is readily applicable for assessing athletic and offensive efficiency, but not defensive efficiency as explained below, it has not to the best of our knowledge been used yet for such purposes in soccer. Previous studies estimating either the athletic efficiency of soccer teams during a season, i.e., Espitia-Escuer and Garcia-Cebrian (2004, 2006) and García-

Sánchez (2007) for the Spanish premier league, or their offensive and defensive efficiency, i.e., García-Sánchez (2007) for the Spanish premier league, Rossi *et al.* (2018) for the Italian premier league, and Boscá *et al.* (2009) for both the Spanish and the Italian league, used conventional DEA models and consequently, they do not account for output interdependency. Thus, if one for example calculates teams' total potential goals scored (conceded), he/she will most likely end up with a figure that exceeds (falls short of) their total actual goals conceded (scored). If this is indeed the case, then the offensive and defensive efficiency scores of the inefficient soccer teams are underestimated in the above studies. The same is essentially true for their athletic efficiency estimates.

The objective of this study is twofold: *first*, we adapt the ZSG-DEA model to the case of an output, such as goals conceded, which may more naturally be viewed as a reverse rather than a forward output, in the sense of Lewis and Sexton (2004).[4] According to them, larger (smaller) values reflect lower (higher) achievements for a reverse output and higher (lower) achievements for a forward output, such as for example goals scored. To introduce the necessary modifications to the ZSG-DEA model, we use as a departure point Lewis and Sexton (2004) DEA model with reverse outputs. In particular, we present the analogues of Lins *et al.*'s (2003) proportional and equal output reduction strategies, which in the case of reverse outputs have an expansion orientation, and we provide the means of estimating the relevant ZSG-DEA efficiency scores under different circumstances regarding both the nature of the returns to scale and the number of outputs considered.

The proposed ZSG-DEA model with reverse outputs is applicable in "situations in which organizations … produce outputs for the explicit purpose of limiting the outputs of other organizations" (Lewis and Sexton, 2004, p. 128) and total (across DMUs) output is fixed. This is certainly the case in sports and specifically in assessing the defensive performance of teams, which, besides soccer that is the focus of this paper, may include basketball, baseball, volleyball, handball, hockey, polo, etc. We have similar situations in political campaigning in which the expenses made by a candidate affect the total votes received by all his/her opponents. In addition, it may be a more appropriate modeling choice for undesirable outputs and the allocation of quota permits as in Gomes and Lins (2008) and Feng *et al*. (2019), where pollutant emissions are treated so far as an input.

The second objective of this paper is to use the ZSG-DEA model with a forward and a reverse output for estimating respectively the offensive and defensive efficiency of soccer teams in Greek premier league. To the best of our knowledge, this is the first time that offensive and defensive efficiency of soccer teams is estimated by explicitly taking into account output interdependency. Even in other sports, this has been done only for the Olympic Games (see for example Lins *et al*. (2003) and Bouzidis and Karagiannis (2021)) and baseball (see Collier *et al*. (2011)).

The rest of this paper unfolds as follows: in the next section we present the ZSG-DEA model with forward outputs and the means for estimating the relevant efficiency scores. In the third section, we present the model and the main results regarding the ZSG-DEA model with reverse outputs. Based on these two formulations of the ZSG-DEA model, we present in the fourth section the empirical results related to offensive and defensive efficiency of soccer teams in the Greek premier league. Concluding remarks follow in the last section.

2. The ZSG-DEA model for forward outputs

Lins *et al.* (2003) proposed the output-oriented variable-returns-to-scale (VRS) ZSG-DEA model for forward fixed-sum outputs as:

$$
\begin{array}{lll}
\underset{h^k, \lambda_k^i, \tilde{y}_{kr}^i}{Max} & h^k & \\
s.t. & \sum_{i=1}^{I} \lambda_k^i x_m^i \le x_m^k, & m = 1, \dots, M; \\
& \sum_{i=1}^{I} \lambda_k^i \tilde{y}_{kr}^i \ge h^k y_r^k, & r = 1, \dots, R; \\
& \sum_{i=1}^{I} \lambda_k^i = 1, & \\
& \lambda_k^i \ge 0, & i = 1, \dots, k, \dots, I;
\end{array}
\tag{1}
$$

where h refers to the efficiency score that is greater-than-or-equal-to one, λ to the intensity variables, x to input quantities, y to actual output quantities, $\tilde{y}$ to resultant output quantities, m is used to index inputs, r to index outputs, and i to index DMUs. The main difference between the above model and the relevant conventional DEA model is that $\tilde{y}$ is among the choice variables in (1). One may substitute $\tilde{y}_k^k = y^k + z^k$, where $z^k \ge 0$ is the output gain of the evaluated DMU, and $\tilde{y}_k^i = y^i - l_k^i$, where $l_k^i \ge 0$ is the output loss of each other DMU, into (1) to yield:

$$
\begin{array}{lll}
\underset{h^k,\lambda_k^i,l_{kr}^i,z_r^k}{Max} & h^k & \\
s.t. & \sum_{i\neq k}^{I}\lambda_k^i x_m^i + \lambda_k^k x_m^k \leq x_m^k\,, & m = 1,\dots,M; \\
& \sum_{i\neq k}^{I}\lambda_k^i\left(y_r^i - l_{kr}^i\right) + \lambda_k^k(y_r^k + z_r^k) \geq h^k y_r^k, & r = 1,\dots,R; \\
& \sum_{i=1}^{I}\lambda_k^i = 1, & \\
& \lambda_k^i \geq 0, & i = 1,\dots,k,\dots,I;
\end{array}
\tag{2}
$$

Both (1) and (2) are non-linear models due to the interaction terms between the choice variables in the second set of their constraints. According to Lins *et al*. (2003), one way to deal with the non-linearity of the ZSG-DEA model is to consider explicit forms of strategies for the DMUs to reach the efficient frontier and for this purpose, they proposed two such strategies: the proportional and the equal output reduction strategy.

2.1 *The Proportional Output Reduction Strategy*

In the case of the proportional reduction strategy and as long as a single output is considered, the output gain of the evaluated DMU is given as $z^k = (h^k - 1)y^k$ and the output loss of each other DMU as $l_k^i = y^i z^k / \sum_{i\neq k}^{I} y^i$. Given that l_k^i depends on both z^k and y^i, it differs by i for each k and consequently, there are $I - 1$ output losses in each "run" of the ZSG-DEA model. Notice also that higher values for y^i imply higher values for l_k^i in such a way that the percentage of the output reduction, i.e., $(z^k / \sum_{i\neq k}^{I} y^i) \times 100$, to be the same for each DMU $i \neq k$. By substituting $z^k = (h^k - 1)y^k$ and $l_k^i = y^i z^k / \sum_{i\neq k}^{I} y^i$ into (2), we get:

$$
\begin{array}{lll}
\underset{h^k,\lambda_k^i}{Max} & h^k & \\
s.t. & \sum_{i\neq k}^{I}\lambda_k^i x_m^i + \lambda_k^k x_m^k \leq x_m^k\,, & m = 1,\dots,M; \\
& \sum_{i\neq k}^{I}\lambda_k^i y^i RC_k + \lambda_k^k h^k y^k \geq h^k y^k, & \\
& \sum_{i=1}^{I}\lambda_k^i = 1, & \\
& \lambda_k^i \geq 0, & i = 1,\dots,k,\dots,I;
\end{array}
\tag{3}
$$

where $RC_k = 1 - \left((h^k - 1)y^k / \sum_{i\neq k}^{I} y^i\right)$ is the reduction coefficient in the k^{th} "run" of (3) with $0 < RC_k \leq 1$.

One of Lins *et al.* (2003) important results is that then there is no need to solve (3) as h^k can be computed by means of the proportional output reduction strategy TAT (PR-TAT), which states that

PROPOSITION 1 (PR-TAT, Lins *et al.*, 2003): In the single output case, the target for a DMU to reach the efficiency frontier in a VRS ZSG-DEA proportional output reduction strategy model equals the same target in the conventional VRS DEA model multiplied by the reduction coefficient; that is, $h^k y^k = \theta^k y^k RC_k$ or $h^k = \theta^k RC_k$.

where θ refers to the conventional DEA efficiency score that is greater-than-or-equal-to one.

An illustrative proof of the PR-TAT, advocated by Lins *et al.* (2003), may be given with reference to Figure 1a, where we consider DMUs a, b and k with input-output bundles of respectively (x^a, y^a), (x^b, y^b) and (x^k, y^k) and the corresponding VRS DEA frontier $x^a ab$.[5] To derive the VRS ZSG-DEA frontier $x^a a'b'$ for the proportional reduction strategy, we take aa' from DMU a and bb' from DMU b and we give them to DMU k ($c'k = aa' + bb'$) to make it ZSG-DEA efficient while, at the same time, we ensure that DMUs a and b are ZSG-DEA efficient. Note that, as the above reductions are proportional to DMUs' actual outputs, $RC_k = a'x^a/ax^a = \tilde{y}^a/y^a = b'x^b/bx^b = \tilde{y}^b/y^b$. If we now use the similarity of triangles acd and abe and of triangles $a'c'd'$ and $a'b'e'$, we get $be/cd = ae/ad$ and $b'e'/c'd' = a'e'/a'd' = ae/ad$. From these, in turn, we get $be/cd = b'e'/c'd'$ that may also be written as $(y^b - y^a)/(\theta^k y^k - y^a) = (\tilde{y}^b - \tilde{y}^a)/(h^k y^k - \tilde{y}^a)$. Then, to prove that $h^k = \theta^k RC_k$, we simply substitute $\tilde{y}^b = y^b RC_k$ and $\tilde{y}^a = y^a RC_k$ into the above relationship, where $RC_k = x^k c'/x^k c$ actually refers to the vertical distance between the VRS DEA frontier and the VRS ZSG-DEA frontier at the level of the input of DMU k.

Based on Proposition 1 and the definition of the reduction coefficient, one can compute the ZSG-DEA efficiency score as (Bi *et al.*, 2014; Bouzidis and Karagiannis, 2021):

$$h^k = \frac{\theta^k \sum_{i=1}^{I} y^i}{\sum_{i \neq k}^{I} y^i + \theta^k y^k} \tag{4}$$

using only data on output quantities and the estimated DEA efficiency score. In addition, if $\lambda_k^k = 0$ and there are no slacks, we get from (3):

$$h^k y^k = \sum_{i \neq k}^{I} \lambda_k^i y^i RC_k \quad (5)$$

Then, from Proposition 1 and (5), it follows that

PROPOSITION 2 (BCET, Lins *et al.*, 2003): In the single output case, the VRS ZSG-DEA proportional output reduction strategy model and the conventional VRS DEA model result in the same set of intensity variables.

This holds since $\theta^k y^k = \sum_{i \neq k}^{I} \lambda_k^i y^i$.

Bi *et al*. (2014) have showed that Propositions 1 and 2 can be extended to the case of constant-returns-to-scale (CRS) as long as we are considering a single output. On the other hand, the ZSG-DEA proportional output reduction strategy model with multiple outputs and either VRS or CRS cannot be converted into a linear model but only to a parametric linear model; see Bi *et al*. (2014).

2.2 *The Equal Output Reduction Strategy*

In the case of the equal reduction strategy, z^k is still given as $z^k = (h^k - 1)y^k$ but output loss differs since it is now given as $l_k^i = z^k/(I-1)$. As in this case l_k^i depends solely on z^k, there is only one output loss in each "run" of the ZSG-DEA model and this is the same for each DMU $i \neq k$. On the contrary, the percentage of the output reduction, i.e., $(z^k/(I-1)y^i) \times 100$, is now different for each DMU $i \neq k$ since it is dependent on y^i. By substituting $z^k = (h^k - 1)y^k$ and $l_k^i = z^k/(I-1)$ into (2) and given that $r = 1$, we get:

$$\begin{aligned}
&\underset{h^k, \lambda_k^i}{Max} && h^k && \\
&s.t. && \textstyle\sum_{i \neq k}^{I} \lambda_k^i x_m^i + \lambda_k^k x_m^k \leq x_m^k\,, && m = 1, \dots, M; \\
& && \textstyle\sum_{i \neq k}^{I} \lambda_k^i \left(y^i - o_k\right) + \lambda_k^k h^k y^k \geq h^k y^k, && \\
& && \textstyle\sum_{i=1}^{I} \lambda_k^i = 1, && \\
& && \lambda_k^i \geq 0, && i = 1, \dots, k, \dots, I;
\end{aligned} \quad (6)$$

where $o_k = (h^k - 1)y^k/(I-1)$ is the output loss in the k^{th} "run" of (6).

Bi *et al.* (2014) have showed that in this case also there is no need to solve (6) as we can compute h^k by means of θ^k. Then, the following proposition may be considered as the additive analogue of the Lins *et al*. (2003) TAT, which we refer to as the equal output reduction strategy TAT (ER-TAT):

PROPOSITION 3 (ER-TAT, Bi *et al*., 2014): In the single output case, the target for a DMU to reach the efficiency frontier in a VRS ZSG-DEA equal output reduction strategy model equals the same target in the conventional VRS DEA model less the output loss; that is, $h^k y^k = \theta^k y^k - o_k$ or $h^k = \theta^k - (o_k/y^k)$.

An illustrative proof of the ER-TAT may be given with reference to Figure 1b, which is similar to Figure 1 of Lins *et al.* (2003). There, we consider DMUs a, b and k and the corresponding VRS DEA frontier $x^a ab$. To derive the VRS ZSG-DEA frontier for the equal reduction strategy, we take aa'' from DMU a and bb'' from DMU b ($aa'' = bb''$) and we give them to DMU k to make it ZSG-DEA efficient while, at the same time, we ensure that DMUs a and b are ZSG-DEA efficient. Then, the VRS ZSG-DEA frontier is given as $x^a a'' b''$ reflecting a neutral inwards shift of the VRS DEA frontier. If we now use the similarity of triangles acd and abe and of triangles $a''c''d''$ and $a''b''e''$, we get $be/cd = ae/ad$ and $b''e''/c''d'' = a''e''/a''d'' = ae/ad$. From these, in turn, we get $be/cd = b''e''/c''d''$ that may be rewritten as $(y^b - y^a)/(\theta^k y^k - y^a) = (\tilde{y}^b - \tilde{y}^a)/(h^k y^k - \tilde{y}^a)$. Then, by substituting $\tilde{y}^b = y^b - o_k$ and $\tilde{y}^a = y^a - o_k$ into the above relationship, we can verify that $h^k = \theta^k - (o_k/y^k)$.

Based on Proposition 3 and the definition of the output loss, one can compute the ZSG-DEA efficiency score as (Bi *et al.*, 2014):

$$h^k = \frac{\theta^k (I-1) + 1}{I} \tag{7}$$

In addition, if there are no slacks and $\lambda_k^k = 0$, then from (6) we get:

$$h^k y^k = \sum_{i \neq k}^{I} \lambda_k^i y^i - o_k \tag{8}$$

From Proposition 3 and (8), it follows that the BCET is also applicable in this case:

PROPOSITION 4 (Bi *et al*., 2014): The BCET holds for the VRS ZSG-DEA equal output reduction strategy model with a single output.

In addition, Bi *et al*. (2014) have showed that Propositions 3 and 4 can be extended to the case of multiple outputs as long as the assumption of VRS is maintained. In contrast, Propositions 3 and 4 do not hold in the case of CRS regardless of whether a single or multiple outputs are considered. In this case, however, (6) can be converted into a linear model by an appropriate variable transformation; see Bi *et al*. (2014).[6]

3. The ZSG-DEA model for reverse outputs

The results presented in the previous section are not readily applicable to the case of a *reverse* output. According to Lewis and Sexton (2004), an output is considered as reverse if larger (smaller) values reflect lower (higher) achievements and its larger values are assumed to be technologically possible in the sense of the free disposability axiom (Olesen *et al*., 2015). To develop the ZSG-DEA model for a reverse output, we rely on the output-oriented VRS Lewis and Sexton (2004) DEA model with a single reverse output, which is given as:

$$\begin{aligned}
\underset{\eta^k,\mu_k^i}{Min} \quad & \eta^k \\
s.t. \quad & \textstyle\sum_{i=1}^{I}\mu_k^i x_m^i \leq x_m^k, \qquad m = 1,\dots,M; \\
& \textstyle\sum_{i=1}^{I}\mu_k^i y^i \leq \eta^k y^k, \\
& \textstyle\sum_{i=1}^{I}\mu_k^i = 1, \\
& \mu_k^i \geq 0, \qquad\qquad i = 1,\dots,k,\dots,I;
\end{aligned} \tag{9}$$

where η refers to the efficiency score that ranges between zero and one and μ to the intensity variables. This is a reformulation of the relevant conventional DEA model, in which the objective function has been changed from maximization to minimization and the greater-than-or-equal-to sign in the second constraint has been changed to a less-than-or-equal-to sign. To adjust (9) for the fixity of the total (across DMUs) actual output, we should treat the y's in the left-hand side of its second constraint as resultant outputs. The difference, however, with the case where a forward fixed-sum output is considered is that now $\tilde{y}_k^k = y^k - z^k$ and $\tilde{y}_k^i = y^i + l_k^i$. By substituting these into (9), we obtain the output-oriented VRS ZSG-DEA model with a reverse output as:

$$
\begin{array}{lll}
\underset{d^k, v_k^i, l_k^i, z^k}{Min} & d^k & \\
s.t. & \sum_{i \neq k}^{I} v_k^i x_m^i + v_k^k x_m^k \le x_m^k, & m = 1, \dots, M; \\
& \sum_{i \neq k}^{I} v_k^i (y^i + l_k^i) + v_k^k (y^k - z^k) \le d^k y^k, & \\
& \sum_{i=1}^{I} v_k^i = 1, & \\
& v_k^i \ge 0, & i = 1, \dots, k, \dots, I;
\end{array}
\tag{10}
$$

where d refers to the efficiency score that ranges between zero and one and v to the intensity variables. As in the case of the ZSG-DEA model with a forward output, (10) is non-linear (see its second constraint) and to convert it into a linear model we have to explicitly specify a zero-sum output redistribution strategy. In the reverse output case, however, we are talking about *expansion* instead of reduction strategies.

3.1. *The Proportional Output Expansion Strategy*

Consider first the proportional expansion strategy, according to which the output level that the evaluated DMU gives to become ZSG-DEA efficient is equal to $z^k = (1 - d^k)y^k$ and it is distributed proportionally among all other DMUs as $l_k^i = y^i z^k / \sum_{i \neq k}^{I} y^i$, implying that the percentage of the output expansion is the same for each of them. By substituting these into (10), we get:

$$
\begin{array}{lll}
\underset{d^k, v_k^i}{Min} & d^k & \\
s.t. & \sum_{i \neq k}^{I} v_k^i x_m^i + v_k^k x_m^k \le x_m^k, & m = 1, \dots, M; \\
& \sum_{i \neq k}^{I} v_k^i y^i EC_k + v_k^k d^k y^k \le d^k y^k, & \\
& \sum_{i=1}^{I} v_k^i = 1, & \\
& v_k^i \ge 0, & i = 1, \dots, k, \dots, I;
\end{array}
\tag{11}
$$

where $EC_k = 1 + \left((1 - d^k)y^k / \sum_{i \neq k}^{I} y^i\right)$ is the expansion coefficient in the k^{th} "run" of (11) with $1 \le EC_k < \infty$. As with (3), (11) is non-linear but as we show next there is no need to be solved as we can compute d^k by means of the proportional output expansion strategy TAT (PE-TAT):

PROPOSITION 5 (PE-TAT): In the case of a single reverse output, the target for a DMU to reach the efficiency frontier in a VRS ZSG-DEA proportional output expansion strategy model equals the same target in the Lewis and Sexton (2004) VRS DEA model multiplied by the expansion coefficient; that is, $d^k y^k = \eta^k y^k EC_k$ or $d^k = \eta^k EC_k$.

An illustrative proof of the PE-TAT may be given with reference to Figure 2a, where we consider DMUs a, b and k and the corresponding VRS DEA frontier T_{DEA}. To derive the VRS ZSG-DEA frontier $T_{ZSG\text{-}DEA}$ for the proportional expansion strategy, we take $\hat{c}k = a\hat{a} + b\hat{b}$ units of the reverse output from DMU k to make it ZSG-DEA efficient. These units are given to DMUs a and b ($a\hat{a}$ to DMU a and $b\hat{b}$ to DMU b) in such a way as to ensure that both of them are ZSG-DEA efficient. Note that, as the above expansions are proportional to DMUs' actual outputs, $EC_k = \hat{a}x^a/ax^a = \tilde{y}^a/y^a = \hat{b}x^b/bx^b = \tilde{y}^b/y^b$. If we now use the similarity of triangles bcd and bae and of triangles $\hat{b}\hat{c}\hat{d}$ and $\hat{b}\hat{a}\hat{e}$, we get $ae/cd = be/bd$ and $\hat{a}\hat{e}/\hat{c}\hat{d} = \hat{b}\hat{e}/\hat{b}\hat{d} = be/bd$. These imply that $ae/cd = \hat{a}\hat{e}/\hat{c}\hat{d}$, which may also be written as $(y^b - y^a)/(y^b - \eta^k y^k) = (\tilde{y}^b - \tilde{y}^a)/(\tilde{y}^b - d^k y^k)$. Then, to show that $d^k = \eta^k EC_k$, we simply substitute $\tilde{y}^b = y^b EC_k$ and $\tilde{y}^a = y^a EC_k$ into the above relationship, where $EC_k = x^k\hat{c}/x^k c$ actually refers to the vertical distance between the VRS ZSG-DEA frontier and the VRS DEA frontier at the level of the input of DMU k.

Based on Proposition 5 and the definition of the expansion coefficient, we can compute the relevant ZSG-DEA efficiency score as:

$$d^k = \frac{\eta^k \sum_{i=1}^{I} y^i}{\sum_{i\neq k}^{I} y^i + \eta^k y^k} \tag{12}$$

In addition, if $\nu_k^k = 0$ and there are no slacks, we get from (11):

$$d^k y^k = \sum_{i\neq k}^{I} \nu_k^i y^i EC_k \tag{13}$$

Then, from Proposition 5 and (13), it follows that

PROPOSITION 6: The BCET holds for the VRS ZSG-DEA proportional output expansion strategy model with a single reverse output.

The above results, i.e., Propositions 5 and 6, can easily be extended to the CRS case but they do not hold if we are considering multiple reverse outputs.

3.2 *The Equal Output Expansion Strategy*

In this sub-section, we consider the equal expansion strategy, where $z^k = (1 - d^k)y^k$ but now it is distributed equally among all other DMUs as $l_k^i = z^k/(I-1)$. By substituting these into (10) yields:

$$
\begin{array}{lll}
\underset{d^k, \nu_k^i}{Min} & d^k & \\
s.t. & \sum_{i \neq k}^{I} \nu_k^i x_m^i + \nu_k^k x_m^k \leq x_m^k, & m = 1, \dots, M; \\
& \sum_{i \neq k}^{I} \nu_k^i (y^i + q_k) + \nu_k^k d^k y^k \leq d^k y^k, & \\
& \sum_{i=1}^{I} \nu_k^i = 1, & \\
& \nu_k^i \geq 0, & i = 1, \dots, k, \dots, I;
\end{array}
\tag{14}
$$

where $q_k = (1 - d^k)y^k/(I-1)$ is the output *intake* in the k^{th} "run" of (14). As with (6), (14) is non-linear but there is no need to be solved as we can compute d^k by means of the equal output expansion strategy TAT (EE-TAT):

PROPOSITION 7 (EE-TAT): In the case of a single reverse output, the target for a DMU to reach the efficiency frontier in a VRS ZSG-DEA equal output expansion strategy model equals the same target in the Lewis and Sexton (2004) VRS DEA model plus the output intake; that is, $d^k y^k = \eta^k y^k + q_k$ or $d^k = \eta^k + (q_k/y^k)$.

An illustrative proof of the EE-TAT may be given with reference to Figure 2b, where we consider DMUs a, b and k and the corresponding VRS DEA frontier T_{DEA}. To derive the VRS ZSG-DEA frontier $T_{ZSG\text{-}DEA}$ for the equal expansion strategy, we take $\check{c}k = a\breve{a} + b\breve{b}$ units of the reverse output from DMU k to make it ZSG-DEA efficient. These units are given to DMUs a and b ($a\breve{a}$ to DMU a and $b\breve{b}$ to DMU b) in such a way as to ensure that both of them are ZSG-DEA efficient. Note that, since $a\breve{a} = b\breve{b}$, $T_{ZSG\text{-}DEA}$ reflects a neutral outwards shift of T_{DEA}. If we now use the similarity of triangles bcd and bae and of triangles $\breve{b}\check{c}\check{d}$ and $\breve{b}\breve{a}\check{e}$, we get $ae/cd = be/bd$ and $\breve{a}\check{e}/\check{c}\check{d} = \breve{b}\check{e}/\breve{b}\check{d} = be/bd$. These imply that $ae/cd = \breve{a}\check{e}/\check{c}\check{d}$, which may be rewritten as $(y^b - y^a)/(y^b - \eta^k y^k) = (\tilde{y}^b - \tilde{y}^a)/(\tilde{y}^b - d^k y^k)$. Then, by substituting $\tilde{y}^b = y^b + q_k$ and $\tilde{y}^a = y^a + q_k$ into the above relationship, we can verify that $d^k = \eta^k + (q_k/y^k)$.

Based on Proposition 7 and the definition of the output intake, we can compute the relevant ZSG-DEA efficiency score as:

$$d^k = \frac{\eta^k(I-1)+1}{I} \tag{15}$$

In addition, if there are no slacks and $\nu_k^k = 0$, we get from (14):

$$d^k y^k = \sum_{i \neq k}^{I} \nu_k^i y^i + q_k \tag{16}$$

From Proposition 7 and (16), it follows that

PROPOSITION 8: The BCET holds for the VRS ZSG-DEA equal output expansion strategy model with a single reverse output.

The above results, i.e., Propositions 7 and 8, can be extended to the case of multiple reverse outputs but they do not hold under CRS. In the case of CRS, one can however convert (14) into a linear model by an appropriate variable transformation.

4. Empirical Results

In this section, we illustrate the empirical applicability of the models presented in the previous two sections by using them to estimate teams' offensive and defensive efficiency during a season of the Greek premier soccer league. Notice that previous studies estimating offensive and defensive efficiency in soccer, i.e., Garcia-Sanchez (2007), Boscá *et al*. (2009) and Rossi *et al*. (2018), have not accounted for output interdependency.

For offensive efficiency, we use *goals scored* as a single forward output and three inputs reflecting teams' offensive actions, namely, (*i*) the number of *shots* and *headers* regardless of whether these were on or off goal, (*ii*) the number of *crosses* regardless of whether these reached a teammate or not, and (*iii*) the number of *assists* regardless of whether these turned into a goal or not. For defensive efficiency, we use *goals conceded* as a single reverse output and three inputs reflecting teams' defensive actions, i.e., (*i*) the number of *saves*, (*ii*) the number of *clearances*, and (*iii*) the number of *steals*. See Table 1 for the summary statistics of all the above variables.[7]

As the size of teams' budget does not necessarily reflect their on-field strengths and weaknesses (Boscá *et al.*, 2009), the above game statistics are really useful in the estimation of teams' offensive and defensive efficiency. In these assessments, we assume an output orientation since the intention of soccer teams is to increase their on-field achievements rather than to decrease their on-field effort. In addition, we assume VRS since soccer teams may differ significantly in terms of the size of their budget, of the know-how, experience and personality of their managerial staff, and of their specific ownership and organization structure, infrastructure, fan base/catchment area, history and tradition.[8]

For the assessment of the offensive efficiency during a soccer league season, output interdependency implies that teams' total potential goals scored ($\sum_{k=1}^{I} \theta^k y^k$) should not differ from their total actual goals conceded, which in turn are equal to the total (across teams) actual goals scored ($\sum_{k=1}^{I} y^k$). On the other hand, for the assessment of the defensive efficiency during a soccer league season, output interdependency implies that teams' total potential goals conceded ($\sum_{k=1}^{I} \eta^k y^k$) should not differ from their total actual goals scored/conceded ($\sum_{k=1}^{I} y^k$). However, according to our empirical results obtained from the conventional DEA model and the DEA model with a reverse output (see the second and third column of Tables 2 and 3, respectively), teams' total potential goals scored are greater than their total actual goals scored $\left(\sum_{k=1}^{18} \theta^k y^k \cong 1011 > \sum_{k=1}^{18} y^k = 833\right)$ and teams' total potential goals conceded are smaller than their total actual goals conceded $\left(\sum_{k=1}^{18} \eta^k y^k = 496 < \sum_{k=1}^{18} y^k = 833\right)$.

However, according to our empirical results obtained from both the ZSG-DEA model with a forward output and the ZSG-DEA model with a reverse output (see Tables 2 and 3), the sums over teams of their resultant outputs $\left(\sum_{i=1}^{18} \tilde{y}_k^i\right)$ *always* equal their total actual (offensive or defensive) output implying that output interdependency between goals scored and goals conceded is accounted for in each "run" of the ZSG-DEA models used. It should be noted that, according to Lins *et al*. (2003), the equal reduction strategy results in negative resultant outputs, if one or more output losses are greater than some actual outputs. In our case, however, all resultant outputs are positive since $max_{o_k} = 1.8$ and $min_{y^k} = 17$. On the other hand, both proportional and equal expansion strategies result in a negative resultant output, if the amount of output given by the evaluated DMU to all others is greater than its actual output. In our empirical application, however, this is not the case for any team since $z^k < y^k \; \forall \; k = 1, \ldots, 18$.

According to our findings, the average offensive inefficiency of teams is estimated at 26.4%, 24.5% and 24.9% by respectively the conventional DEA model, the ZSG-DEA proportional reduction strategy model and the ZSG-DEA equal reduction strategy model. From these results, it seems that the conventional DEA model overestimates teams' average offensive inefficiency by 1.5-2%. However, according to the rank correlation coefficients reported in the lower part of Table 2, there are no rank differences in teams' offensive efficiency scores obtained by the above three models. On the other hand, the average defensive inefficiency of teams is computed at 36.2%, 35.1% and 34.2% by respectively the DEA model with a reverse output, the ZSG-DEA proportional expansion strategy model and the ZSG-DEA equal expansion strategy model. From these results, it seems that the DEA model with a reverse output overestimates teams' average defensive inefficiency by 1-2%. However, according to the rank correlation coefficients reported in the lower part of Table 3, there are no rank differences in teams' defensive efficiency scores obtained by the aforementioned three models. Since their average offensive efficiency is greater than their average defensive efficiency, performance heterogeneity among teams is higher in defense than it is in offense. In other words, it may be said that the major deficit of teams during the soccer league season under consideration is their performance in defense.

As it was expected due to the BCET, all three models used for the estimation of the offensive (defensive) efficiency result in the same set of efficient teams, i.e., #1, #5, #10, #12, #14, #15 and #18 (#1, #3, #7 and #10). Since most of the offensively efficient teams are in the lower part of the final league ranking that is based on teams' accumulated points, we may argue that being efficient in offense did not necessarily guarantee a high enough points' accumulation. On the other hand, most of the defensively efficient teams are in the upper part of the final league ranking signifying the importance of defensive efficiency in points' accumulation. Notice here that only teams #1 and #10 are both offensively and defensively efficient and, even though there is no doubt for the exceptional on-field performance of the champion, this is quite an interesting result for team #10 given that it lies away from the top of the final league ranking, implying that its points' accumulation is possibly ineffective. That is, being efficient in offense and defense might be necessary but not sufficient for winning the championship. In addition, it follows that top-ranked teams, i.e., #2 to #6, could earn crucial points even when they do not perform efficiently in offense and/or defense. In other words, they are capable of accumulating the greatest number of points given a determined offensive and defensive efficiency level (Garcia-Sanchez, 2007) since they probably employ players that are experienced in competing under pressure and winning close games.

By comparing now the results from the ZSG-DEA equal reduction strategy model and the ZSG-DEA proportional reduction strategy model, we see that (with the exception of teams #11 and #16) the offensive efficiency scores resulted from the latter model are greater-than-or-equal-to those resulted from the former model. However, the opposite is true for the estimated output gains. On the other hand, if we compare the results from the ZSG-DEA equal expansion strategy model and the ZSG-DEA proportional expansion strategy model, we notice that all the defensive efficiency scores obtained from the former model are greater-than-or-equal-to those obtained from the latter model.

To examine further the differences between the results for defensive efficiency obtained from the ZSG-DEA model with a reverse output and models not accounting for output interdependency, we also estimate the conventional DEA model using two commonly employed data transformation approaches. The first is based on the inverse of goals conceded, as in Garcia-Sanchez (2007), and the second on the min-max transformation of goals conceded, i.e., $w^k = (\max_k y^k - y^k)/(\max_k y^k - \min_k y^k)$ (see e.g. Wang (1997)).[9] The relevant results are reported in Tables 4 and 5. From the values of the rank correlation coefficients reported in the lower panel of these Tables, we can see that there are no rank differences in the defensive efficiency scores among the different models.

5. Concluding Remarks

In this paper, we adapted the ZSG-DEA model to the case of a reverse output in the sense of Lewis and Sexton (2004), whose larger (smaller) values reflect lower (higher) achievements. Then, we introduced the zero-sum reverse output redistribution strategies and specified the expansion coefficient and output intake terms. In addition, we stated the resulting TAT for the VRS proportional expansion strategy model with a single reverse output. According to this theorem, there is a multiplicative relationship between the potential output of each DMU evaluated by the above model and its potential output obtained from the Lewis and Sexton (2004) VRS DEA model with a single reverse output. Further, we stated the resulting TAT also for the VRS equal expansion strategy model with a single reverse output. According to this theorem, there is an additive relationship between the potential output of each DMU evaluated by the above model and its potential output obtained from the Lewis and Sexton (2004) VRS DEA model with a single reverse output. Besides, we confirmed that, in the single reverse output case and under VRS, the BCET is also applicable to both the proportional and the equal expansion strategy model. Next, we provided the means of estimating the relevant ZSG-DEA efficiency scores under different circumstances regarding both the nature of the returns to scale and the number of outputs considered. Lastly, we used the ZSG-DEA model with a forward and a reverse output for estimating respectively the offensive and defensive efficiency of teams during a season of the Greek premier soccer league. In this way, we actually showed three things: *first*, how goals conceded can be treated as the reverse fixed-sum output of defense; *second*, how the output interdependency between goals scored and goals conceded can be accommodated into the DEA models used in soccer; *three*, how to estimate teams' ZSG-DEA efficiency scores in offense and defense directly from their corresponding DEA efficiency scores. These could be useful for the future empirical applications in this field since previous ones, such as those presented in García-Sánchez (2007), Boscá *et al.* (2009) and Rossi *et al.* (2018), are based on conventional DEA models that do not account for the output interdependency in offense and defense and as a consequence, they probably report offensive and defensive efficiency scores that should be handled with caution.

From our findings in the league season under consideration, it may be concluded that (*i*) the major deficit of teams is their performance in defense, (*ii*) being efficient in offense does not necessarily guarantee a high enough points' accumulation, (*iii*) the defensive efficiency of teams is very important for their points' accumulation, (*iv*) being efficient in offense and defense might be a necessary but not sufficient condition for winning the championship, (*v*) top-ranked teams could earn crucial points even when they do not perform efficiently in offense and/or defense, (*vi*) average offensive inefficiency is overestimated by the conventional DEA model, and (*vii*) average defensive inefficiency is overestimated by the DEA model with a reverse output. Nevertheless, for the data at hand, we have found no rank differences in the offensive efficiency scores estimated by the conventional DEA model, the ZSG-DEA proportional reduction strategy model, and the ZSG-DEA equal reduction strategy model. The same holds also for the defensive efficiency scores computed by the DEA model with a reverse output, the ZSG-DEA proportional expansion strategy model, and the ZSG-DEA equal expansion strategy model.

Conflict of Interest:

The authors declare that they have no conflict of interest.

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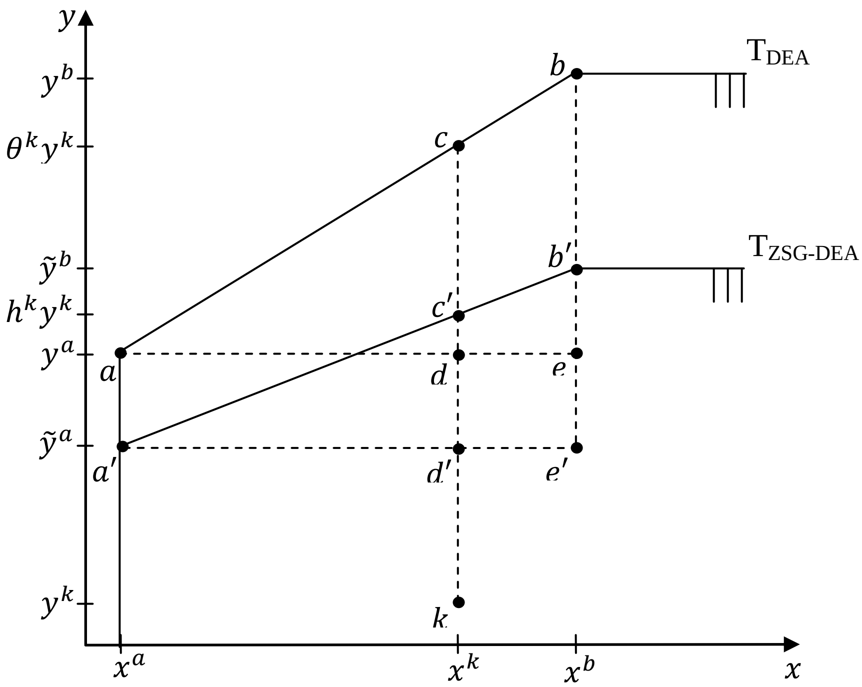
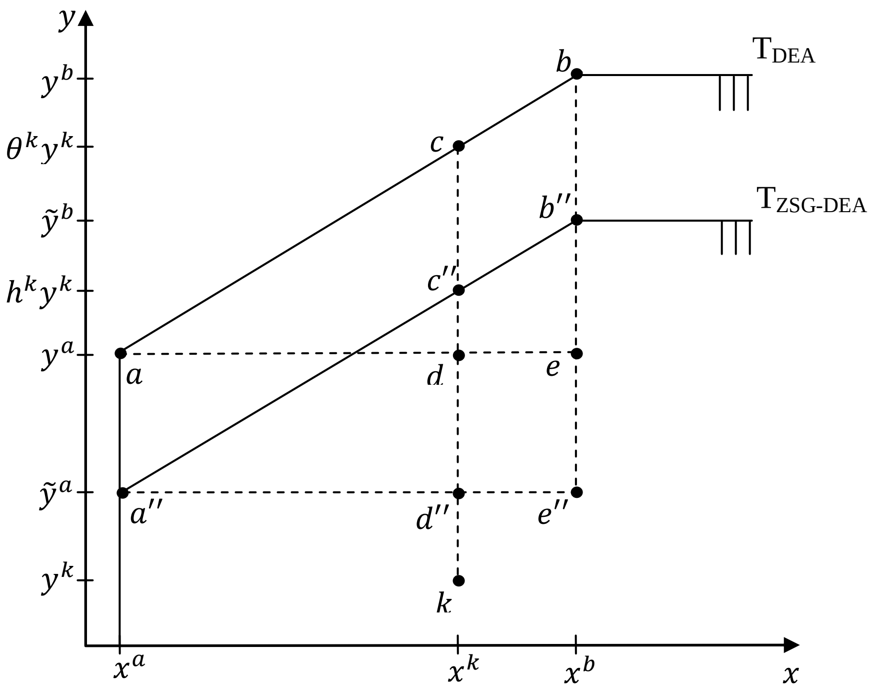
Figure 1: Conventional DEA and ZSG-DEA frontiers for a forward output

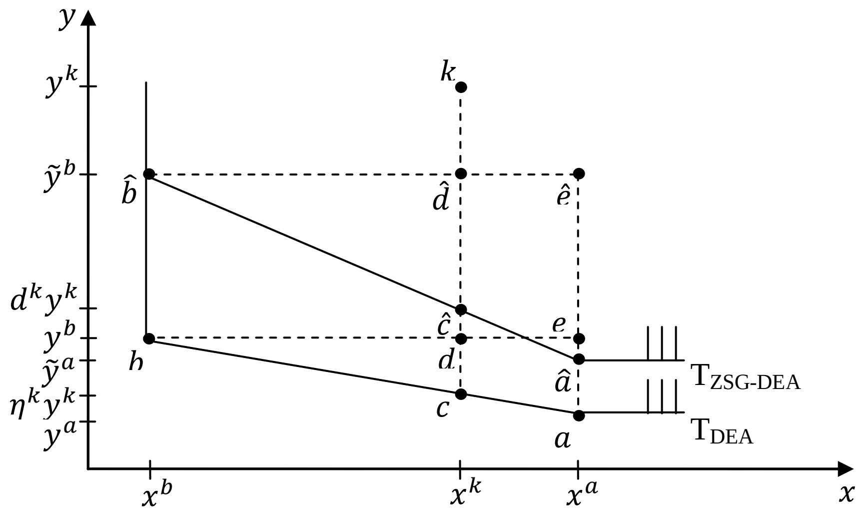


(a): Proportional reduction strategy

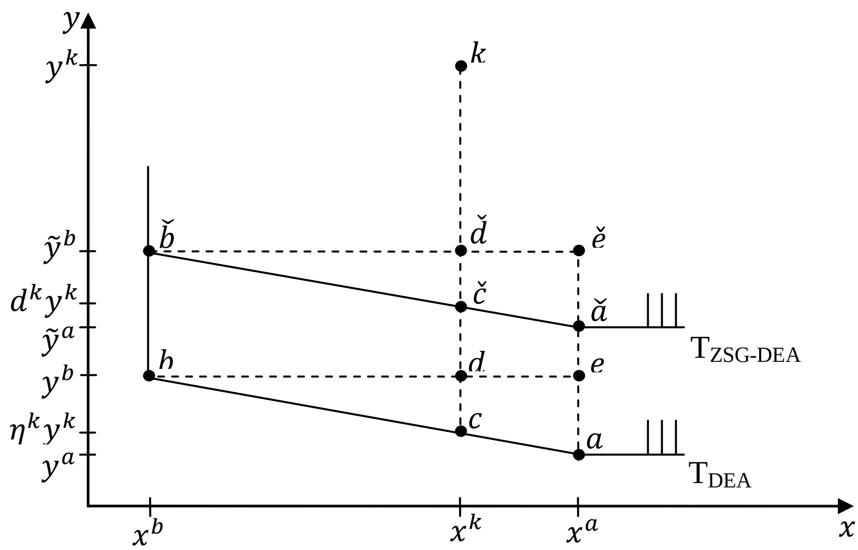


(b): Equal reduction strategy

Figure 2: DEA and ZSG-DEA frontiers for a reverse output



(a): Proportional expansion strategy



(b): Equal expansion strategy

Table 1: Summary Statistics, 1998-99 Greek Premier Soccer League

Final Ranking	Offense				Defense			
Teams	Shots and Headers	Crosses	Assists	Goals Scored	Saves	Clearances	Steals	Goals Conceded
1. OSFP	596	1081	65	82	125	438	484	22
2. AEK	594	1144	74	71	148	449	552	27
3. PAO	726	1308	81	66	155	576	446	36
4. PAOK	623	1163	91	52	169	563	483	31
5. IONIKOS	541	1027	40	64	203	731	478	36
6. ARIS	554	876	69	53	208	650	506	43
7. SKODA XANTHI	515	881	43	44	216	641	411	33
8. OFI	547	1392	61	50	164	797	590	44
9. IRAKLIS	558	930	66	54	200	695	547	45
10. KAVALA	438	698	41	46	246	802	392	62
11. ETHNIKOS ASTIR	433	778	31	40	271	740	420	58
12. PROODEFTIKI	351	784	23	28	265	820	564	37
13. PANILIAKOS	450	855	30	37	243	838	434	54
14. APOLLON ATHENS	389	825	24	42	212	707	436	62
15. PANIONIOS	394	785	28	42	233	643	458	58
16. PANELEFSINIAKOS	426	771	25	25	259	776	488	49
17. VERIA	480	889	35	20	249	746	450	55
18. ETHNIKOS PIRAEUS	341	593	15	17	330	921	481	81
Sum	8956	16780	842	833	3896	12533	8620	833
Max	726	1392	91	82	330	921	590	81
Min	341	593	15	17	125	438	392	22
Average	497.6	932.2	46.8	46.3	216.4	696.3	478.9	46.3
Standard Deviation	103.6	212.9	22.9	17.6	51.6	130	55.5	15

Source: *Galanis Sports Data* (www.galanissportsdata.com)

Table 2: Offensive Efficiency, 1998-99 Greek Premier Soccer League

Teams	Conventional DEA		ZSG-DEA Proportional Reduction Strategy				ZSG-DEA Equal Reduction Strategy			
k	θ^k	$\theta^k y^k$	h^k	z^k	$\sum_{i=1}^{18} \tilde{y}_k^i$	RC_k	h^k	z^k	$\sum_{i=1}^{18} \tilde{y}_k^i$	o_k
1	1	82	1	0	833	1	1	0	833	0
2	1.149	81.6	1.135	9.6	833	0.987	1.141	10	833	0.6
3	1.242	82	1.219	14.4	833	0.981	1.229	15.1	833	0.9
4	1.577	82	1.522	27.2	833	0.965	1.545	28.4	833	1.7
5	1	64	1	0	833	1	1	0	833	0
6	1.183	62.7	1.170	9	833	0.988	1.173	9.2	833	0.5
7	1.307	57.5	1.286	12.6	833	0.984	1.290	12.8	833	0.8
8	1.451	72.6	1.413	20.7	833	0.974	1.426	21.3	833	1.3
9	1.256	67.8	1.236	12.7	833	0.984	1.242	13.1	833	0.8
10	1	46	1	0	833	1	1	0	833	0
11	1.099	44	1.094	3.7	833	0.995	1.093	3.7	833	0.2
12	1	28	1	0	833	1	1	0	833	0
13	1.289	47.7	1.272	10.1	833	0.987	1.273	10.1	833	0.6
14	1	42	1	0	833	1	1	0	833	0
15	1	42	1	0	833	1	1	0	833	0
16	1.548	38.7	1.523	13.1	833	0.984	1.518	12.9	833	0.8
17	2.646	52.9	2.545	30.9	833	0.962	2.554	31.1	833	1.8
18	1	17	1	0	833	1	1	0	833	0
Sum		1010.5								
Average	1.264	56.1	1.245	9.1	833	0.988	1.249	9.3	833	0.5
Standard Deviation	0.396	19.6	0.371	9.7	-	0.012	0.374	10	-	0.6

Rank Correlation Coefficients	ZSG-DEA Proportional Reduction Strategy	ZSG-DEA Equal Reduction Strategy
Conventional DEA	0.998	1
ZSG-DEA Proportional Reduction Strategy		0.998

Table 3: Defensive Efficiency, 1998-99 Greek Premier Soccer League

Teams	DEA with a reverse output		ZSG-DEA							
			Proportional Expansion Strategy				Equal Expansion Strategy			
k	η^k	$\eta^k y^k$	d^k	z^k	$\sum_{i=1}^{18} \tilde{y}_k^i$	EC_k	d^k	z^k	$\sum_{i=1}^{18} \tilde{y}_k^i$	q_k
1	1	22	1	0	833	1	1	0	833	0
2	0.815	22	0.820	4.9	833	1.006	0.825	4.7	833	0.3
3	1	36	1	0	833	1	1	0	833	0
4	0.715	22.2	0.722	8.6	833	1.011	0.730	8.4	833	0.5
5	0.636	22.9	0.646	12.7	833	1.016	0.656	12.4	833	0.7
6	0.512	22	0.525	20.4	833	1.026	0.539	19.8	833	1.2
7	1	33	1	0	833	1	1	0	833	0
8	0.500	22	0.514	21.4	833	1.027	0.528	20.8	833	1.2
9	0.489	22	0.503	22.4	833	1.028	0.517	21.7	833	1.3
10	1	62	1	0	833	1	1	0	833	0
11	0.546	31.6	0.563	25.3	833	1.033	0.571	24.9	833	1.5
12	0.595	22	0.605	14.6	833	1.018	0.617	14.2	833	0.8
13	0.547	29.5	0.563	23.6	833	1.030	0.572	23.1	833	1.4
14	0.471	29.2	0.491	31.6	833	1.041	0.501	30.9	833	1.8
15	0.447	25.9	0.465	31	833	1.040	0.478	30.3	833	1.8
16	0.449	22	0.464	26.3	833	1.033	0.480	25.5	833	1.5
17	0.493	27.1	0.510	26.9	833	1.035	0.521	26.3	833	1.5
18	0.277	22.5	0.298	56.9	833	1.076	0.317	55.3	833	3.3
Sum		496								
Average	0.638	27.6	0.649	18.1	833	1.023	0.658	17.7	833	1
Standard Deviation	0.229	9.7	0.222	14.8	-	0.019	0.216	14.4	-	0.8

Rank Correlation Coefficients	ZSG-DEA Proportional Expansion Strategy	ZSG-DEA Equal Expansion Strategy
DEA with a reverse output	0.998	1
ZSG-DEA Proportional Expansion Strategy		0.998

Table 4: Defensive Efficiency, 1998-99 Greek Premier Soccer League

Teams	Conventional DEA Model with a Transformed Forward Output	ZSG-DEA Model with a Transformed Forward Output: Proportional Reduction Strategy	ZSG-DEA Model with a Transformed Forward Output: Equal Reduction Strategy
k	θ_1^k	h_1^k	h_1^k
1	1	1	1
2	1.227	1.204	1.214
3	1	1	1
4	1.403	1.362	1.380
5	1.592	1.534	1.559
6	1.953	1.858	1.900
7	1	1	1
8	2.000	1.900	1.944
9	2.045	1.941	1.987
10	1	1	1
11	1.866	1.803	1.818
12	1.681	1.612	1.643
13	1.894	1.824	1.844
14	2.203	2.108	2.136
15	2.326	2.209	2.252
16	2.227	2.105	2.159
17	2.110	2.016	2.048
18	3.636	3.382	3.490
Average	1.787	1.714	1.743
Standard Deviation	0.656	0.595	0.620

Rank Correlation Coefficients	Proportional Reduction Strategy Model	Equal Reduction Strategy Model
Conventional DEA Model with a Transformed Forward Output	0.998	1
Proportional Reduction Strategy Model		0.998
	DEA Model with a Reverse Output	
Conventional DEA Model with a Transformed Forward Output	0.996	
	ZSG-DEA Proportional Expansion Strategy Model with a Reverse Output	
ZSG-DEA Proportional Reduction Strategy Model with a Transformed Forward Output	0.990	
	ZSG-DEA Equal Expansion Strategy Model with a Reverse Output	
ZSG-DEA Equal Reduction Strategy Model with a Transformed Forward Output	0.996	

Note: The transformed forward output is the inverse of teams' goals conceded.

Table 5: Defensive Efficiency, 1998-99 Greek Premier Soccer League

Teams	Conventional DEA Model with a Transformed Forward Output	ZSG-DEA Model with a Transformed Forward Output: Proportional Reduction Strategy	ZSG-DEA Model with a Transformed Forward Output: Equal Reduction Strategy
k	θ_2^k	h_2^k	h_2^k
1	1	1	1
2	1.093	1.084	1.088
3	1	1	1
4	1.178	1.161	1.168
5	1.290	1.264	1.274
6	1.553	1.502	1.522
7	1	1	1
8	1.595	1.541	1.562
9	1.639	1.581	1.604
10	1	1	1
11	2.146	2.059	2.082
12	1.340	1.309	1.322
13	1.905	1.833	1.854
14	2.725	2.589	2.629
15	2.392	2.276	2.315
16	1.845	1.769	1.798
17	2.070	1.982	2.011
18	-	-	-
Average	1.575	1.526	1.543
Standard Deviation	0.536	0.493	0.506

Rank Correlation Coefficients	Proportional Reduction Strategy Model	Equal Reduction Strategy Model
Conventional DEA Model with a Transformed Forward Output	1	1
Proportional Reduction Strategy Model		1

	DEA Model with a Reverse Output
Conventional DEA Model with a Transformed Forward Output	0.917
	ZSG-DEA Proportional Expansion Strategy Model with a Reverse Output
ZSG-DEA Proportional Reduction Strategy Model with a Transformed Forward Output	0.908
	ZSG-DEA Equal Expansion Strategy Model with a Reverse Output
ZSG-DEA Equal Reduction Strategy Model with a Transformed Forward Output	0.917

Note: The transformed forward output is $w^k = \left(max_{GC^k} - GC^k\right)/\left(max_{GC^k} - min_{GC^k}\right)$ where GC refers to teams' goals conceded.

Footnotes

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[1] The "runs" of the ZSG-DEA model are as many as the number of the evaluated DMUs.

[2] These are related to the number of the outputs considered, to the assumed nature of the returns to scale, and to the adopted reduction strategy.

[3] Gomes and Lins (2008) coined these names given that, in Lins *et al*. (2003), the TAT and the BCET are simply referred to as Theorem and Corollary, respectively.

[4] Olesen *et al*. (2015) also referred to this type of outputs and they mentioned that if larger values are possible in the sense of free disposability axiom then in an activity analysis model they are modeled by means of inequalities similar to those used for inputs. This is essentially what Lewis and Sexton (2004) did in their formulation of the DEA model with reverse outputs. Notice also that Liu *et al*. (2010) referred to the free disposability axiom for both forward and reverse outputs as extended strong disposability.

[5] Figure 1a is similar to Figure 2 in Lins *et al.* (2003), where a formal proof of the PR-TAT is also provided.

[6] This linearized form of (6) is also applicable under VRS but as Propositions 3 and 4 hold in this case, there is no need to estimate such a model.

[7] For the selection of the considered variables, we are based on García-Sánchez (2007), Boscá *et al.* (2009) and Rossi *et al.* (2018), who also estimate teams' offensive and defensive efficiency during a soccer league season but without taking the output interdependency between goals scored and goals conceded into account. Specifically, all the above studies use the number of goals scored as a single offensive output while, for the offensive inputs, García-Sánchez (2007) considers the number of attacking moves, the number of passes to the opponents' penalty area and the number of shots on goal, Boscá *et al.* (2009) the number of shots on goal, the number of attacking moves, the number of balls kicked into the opponents' centre area and the minutes of ball possession, and Rossi *et al.* (2018) the number of shots, the number of counter attacks, the number of crosses completed, the number of passes completed and the number of useful dribbles. On the other hand, the aforementioned studies use the *inverse* of the number of goals conceded as a single defensive output while, for the defensive inputs, García-Sánchez (2007) considers the number of ball recoveries and

the number of goalkeeper's actions, Boscá *et al.* (2009) the inverse of the number of shots on goal made by opponents, the inverse of the number of attacking moves made by opponents in the goal area, the inverse of the number of passes made by opponents to the centre area, and the inverse of the minutes of ball possession by opponents, and Rossi *et al.* (2018) the number of saves, the number of anticipations, the number of tackles, the number of clearances and the number of times that opponents have been offside.

[8] Boscá *et al.* (2009) and Rossi *et al.* (2018) employ an output-oriented CRS conventional DEA model while García-Sánchez (2007) uses an output-oriented conventional DEA model that adopts both CRS and VRS.

[9] Notice however that their use is associated with theoretical problems: the former as a ratio output is inconsistent with the convexity assumption (Emrouznejad and Amin, 2009) and the latter is inconsistent with translation invariance.