On VEA, production trade-offs and weights restrictions

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ABSTRACT: In this paper we explore the relationship between Value Efficiency Analysis (VEA) and Data Envelopment Analysis (DEA) models including production trade-offs or weights restrictions. In particular, we show that the VEA model is equivalent to a DEA model including production trade-offs for which the trade-off coefficient vectors are equal to either (i) the negative of the input and output quantities of the Decision Making Units (DMUs) chosen as the Most Preferred Solution (MPS) in VEA, under constant returns to scale, or (ii) the deviations of all evaluated DMUs' input and output quantities from those of the DMUs chosen as the MPS, irrespectively of the returns-to-scale assumption. These trade-offs are the dual forms of type II Assurance Region weight restrictions. We then show that a similar equivalence holds between pure output or input VEA and DEA models including trade-offs, if the above trade-offs are respectively considered only for outputs or only for inputs. In this case the trade-offs are the dual forms of type I Assurance Region weight restrictions.

KEYWORDS: Data Envelopment Analysis, Value Efficiency Analysis, Weight Restrictions, Production Trade-Offs.

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1. Introduction

Production trade-offs, their dual weights restrictions, and Value Efficiency Analysis (VEA) are alternative ways of incorporating preference information in Data Envelopment Analysis (DEA). In particular, Decision Maker's (DM) preferences are used to restrict the admissible values of the input/output multipliers in a DEA model. Production trade-offs reflect acceptable marginal changes in inputs and/or outputs that modify their target values for each evaluated Decision Making Unit (DMU) in the envelopment form of DEA models (Podinovski, 2004). Their dual counterpart are the well-known weights restrictions, namely additional linear inequalities in the multiplier form of DEA models that restrict the flexibility of input/output weights based on DM's knowledge, value judgements or in general, holding views for their relative importance (see e.g., Allen *et al*, 1997). DEA models including production trade-offs have been used, among others, for the assessment of efficiency in healthcare (Amado and Dyson, 2009), education (Khalili *et al.*, 2010a), electricity distributors (Santos *et al.*, 2011), and farmers (Atici and Podinovski, 2015). On the other hand, in VEA, the performance of each DMU is assessed relative to the Most Preferred Solution (MPS), namely a non-dominated (i.e., efficient) DMU or a combination of DMUs that has the most desirable input/output bundle by view of a DM or reflects DM's preferences about input/output mixes (Halme *et al.*, 1999). In such a case, each DMU's input/output weights are restricted to values among only those that are optimal for the

MPS in DEA. This in turn results in extending the DEA efficient facets generated by it. Recent applications of VEA include, but are not limited to, the evaluation of hospital departments (Halme and Korhonen, 2000), academic institutions (Korhonen *et al.,* 2001), retail stores (Korhonen *et al.,* 2002) as well as bank branches (Halme *et al.,* 2014).

Several studies in the literature have examined the effect of including production trade-offs or their dual weight restrictions in DEA models. For example, Podinovski (2005) demonstrated the effects of using additional restrictions such as weight bounds in the evaluation results of DEA models, Asmild *et al.* (2006) investigated the potential relation of DEA models with trade-offs to models assessing economic (i.e., cost, revenue, or profit) efficiency, and Podinovski (2007a) developed a procedure for obtaining efficient targets in DEA models with production trade-offs. Also, Podinovski and Forsund (2010) assessed the effects of introducing production trade-offs in the quantitative and qualitative returns-to-scale estimates of DMUs, while Podinovski and Bouzdine-Chameeva (2013) developed linear programs for testing whether the use of a particular set of production trade-offs in DEA models results in violating production assumptions. On the other hand, the similarities between VEA and various forms of weight restrictions have been noted in the literature, but not yet thoroughly examined. For instance, Sarrico and Dyson (2004, p. 18) considered VEA as ‘’another alternative to incorporating the decision maker's preferences into the assessment of DMUs’’, while Kao and Hung (2005, p. 1197) noted that VEA is ‘’essentially an approach of weight restrictions’’. Angulo-Meza and Estellita-Lins (2002, p. 225) viewed VEA and weights restrictions as methodologies incorporating "information provided by a decision maker or expert into the model", while Adler *et al.* (2002) referred to VEA as one of the methods that use "preference information to further refine the discriminatory power of DEA models".[1] Nevertheless, none of these studies have explicitly related VEA to DEA models including weights restrictions, as well as their dual production trade-offs. Such explicit relationships, if any, have, to the best of our knowledge, not yet been investigated.

The purpose of this paper is to explore the relation between VEA and DEA models including production trade-offs and their dual weights restrictions in a detailed manner. More specifically, we show that, under constant returns to scale, the VEA model is equivalent to a DEA model including production trade-offs, for which the trade-off coefficient vectors are given by the negative of the input and output quantities of the DMUs chosen as the MPS in VEA. We also show that, regardless of the nature of returns to scale, the VEA model is equivalent to a DEA model with production trade-offs, for which the trade-offs coefficient vectors are given by the deviations of every DMU's input and output quantities from those of the MPS. These production trade-offs result in extending certain facets of the DEA frontier and in both cases are dual to Type II Assurance Region (AR-II) weight restrictions (see Thompson *et al.,* 1990). Considering the above trade-offs only for the inputs or the outputs we can prove a similar equivalence between pure input or output VEA and DEA models. The dual form of these trade-offs, which refer only to the inputs or the outputs, are type I Assurance Region (AR-I) weight restrictions (Thompson *et al.*, 1986).

The rest of the paper unfolds as follows: In the second section we discuss VEA and DEA models with production trade-offs. The papers' main results are presented in the third section, while an empirical application follows in the fourth section. Concluding remarks follow in the last section.

2. Materials and methods

Production trade-offs are the dual form of weights restrictions that are usually appended in the multiplier form of DEA models. They refer to marginal changes between inputs and/or outputs that take place at some point at the conventional DEA frontier and enlarge the feasible space with additional input/output possibilities (Podinovski, 2004). These changes represent perceptions regarding the normative substitution rates between inputs or transformation rates between outputs, or simply judgements about the relative importance of different inputs and outputs. They are considered as acceptable by all evaluated DMUs, in the sense that it is unanimously agreed that they result in feasible (technologically possible) input/output combinations. Then, one may argue that the targets identified for inefficient DMUs on the enlarged parts of the DEA frontier are in principle technologically realistic or feasible (Podinovski, 2007b).

Let us consider a set of K DMUs $(k=1,\dots,K)$ using the same technology and producing a set of J $(j=1,\dots,J)$ outputs utilizing I $(i=1,\dots,I)$ inputs. Assume further that there exists a set of R $(r=1,\dots,R)$ trade-off relations among inputs and/or outputs, which may be represented as:

$$P^r = [p_1^r, \dots, p_i^r, \dots, p_I^r]^T, \;\; r = 1, \dots, R$$
$$Q^r = \left[q_1^r, \dots, q_j^r, \dots, q_J^r\right]^T, \;\; r = 1, \dots, R \qquad (1)$$

Each of the trade-offs in (1) refers to an agreed postulate among DMUs that by changing the level of each of a DMU's inputs by the trade-off coefficient p_i^r and each of its outputs by the trade-off coefficient q_j^r results in a new unobserved input/output combination that is feasible. Thus, the vectors P^r and Q^r modify respectively the target values of inputs and outputs in the envelopment form of a DEA model, which in turn results in enlarging the DEA efficient frontier with additional linear segments, i.e., facets.

The multiplier and envelopment form of an input-oriented, constant returns to scale (CRS) DEA model including trade-offs as in (1) is given as (Podinovski, 2004):[2]

$$\begin{aligned}
&\max_{u_j^k, v_i^k} \sum_{j=1}^{J} u_j^k y_j^k \\
&s.t. \;\; \sum_{j=1}^{J} u_j^k y_j^h - \sum_{i=1}^{I} v_i^k x_i^h \le 0 \quad && h = 1, \dots, K \\
&\quad\;\; \sum_{j=1}^{J} u_j^k q_j^r - \sum_{i=1}^{I} v_i^k p_i^r \le 0 \quad && r = 1, \dots, R \\
&\quad\;\; \sum_{i=1}^{I} v_i^k x_i^k = 1 \\
&\quad\;\; u_j^k \ge 0 && j = 1, \dots, J \\
&\quad\;\; v_i^k \ge 0 && i = 1, \dots, I
\end{aligned}
\qquad
\begin{aligned}
&\min_{\theta_{TO}^k, \lambda_h^k} \theta_{TO}^k \\
&s.t. \;\; \sum_{h=1}^{K} \lambda_h^k y_j^h + \sum_{r=1}^{R} \pi_r^k q_j^r \ge y_j^k \quad && j = 1, \dots, J \\
&\quad\;\; \sum_{h=1}^{K} \lambda_h^k x_i^h + \sum_{r=1}^{R} \pi_r^k p_i^r \le \theta_{TO}^k x_i^k \quad && i = 1, \dots, I \\
&\quad\;\; \lambda_h^k \ge 0 && h = 1, \dots, K \\
&\quad\;\; \pi_r^k \ge 0 && r = 1, \dots, R \\
&\quad\;\; \theta_{TO}^k \; free
\end{aligned}
\qquad (2)$$

where *x* and *y* are respectively the quantities of inputs and outputs, *v* and *u* are their input and output weights, θ is the efficiency score, λ are the intensity variables, and π are the proportions by which each of the trade-offs is applied to modify the input and output targets. On the other hand, the variable returns to scale (VRS) counterpart of (2) is given as:

$$\max_{u_j^k, v_i^k, u^k} \sum_{j=1}^{J} u_j^k y_j^k + u^k$$

$$\begin{aligned}
s.t.\quad & \sum_{j=1}^{J} u_j^k y_j^h - \sum_{i=1}^{I} v_i^k x_i^h + u^k \le 0 && h = 1, \dots, K \\
& \sum_{j=1}^{J} u_j^k q_j^r - \sum_{i=1}^{I} v_i^k p_i^r \le 0 && r = 1, \dots, R \\
& \sum_{i=1}^{I} v_i^k x_i^k = 1 \\
& u_j^k \ge 0 && j = 1, \dots, J \\
& v_i^k \ge 0 && i = 1, \dots, I \\
& u^k \ free
\end{aligned}$$

$$\min_{\theta_{TO}^k, \lambda_h^k} \theta_{TO}^k$$

$$\begin{aligned}
s.t.\quad & \sum_{h=1}^{K} \lambda_h^k y_j^h + \sum_{r=1}^{R} \pi_r^k q_j^r \ge y_j^k && j = 1, \dots, J \\
& \sum_{h=1}^{K} \lambda_h^k x_i^h + \sum_{r=1}^{R} \pi_r^k p_i^r \le \theta_{TO}^k x_i^k && i = 1, \dots, I \\
& \sum_{h=1}^{K} \lambda_h^k = 1 \\
& \lambda_h^k \ge 0 && h = 1, \dots, K \\
& \pi_r^k \ge 0 && r = 1, \dots, R \\
& \theta_{TO}^k \ free
\end{aligned} \tag{3}$$

in which the free variable u^k is dual to the convexity constraint in the envelopment form of (3).

From (2) and (3) we can see that incorporation of trade-offs such as in (1) into the envelopment form of the DEA model is equivalent to including the following set of homogeneous weight restrictions in its multiplier form:[3]

$$\sum_{j=1}^{J} u_j^k q_j^r - \sum_{i=1}^{I} v_i^k p_i^r \le 0, \ \ r = 1, \dots, R \tag{4}$$

The weight restrictions in (4) concern value judgments regarding (i) only inputs, if $q_j^r = 0$ for $j = 1, \dots, J$, (ii) only outputs, if $p_i^r = 0$ for $i = 1, \dots, I$, or (iii) both inputs and outputs, if $q_j^r \ne 0$ for at least one $j = 1, \dots, J$ and $p_i^r \ne 0$ for at least one $i = 1, \dots, I$.[4] In the former two cases they are referred to as AR-I (Thompson *et al.*, 1986), while in the latter case as AR-II weight restrictions (Thompson *et al.*, 1990).

On the other hand, in VEA, a DM expresses his/her preferences over the desirable input/output bundle or mix by choosing a DMU or a combination of DMUs as the MPS (Halme *et al.*, 1999). This might be a more appealing way of expressing preferences, as DMs are usually more keen to choose desirable values for the inputs and the outputs rather that weight bounds (Korhonen *et al.*, 2002). The VEA frontier is then constructed as the lower envelope of the extended DEA efficient facets intercepting at the MPS. As the facets of the DEA efficient frontier are generated by extreme-efficient DMUs, the MPS will in essence be either a single extreme-efficient DMU or a combination of extreme-efficient DMUs that are *jointly* efficient, in the sence that they generate at least one common facet.[5] VEA then extends only these common facets among the DMUs comprising the MPS.

The input-oriented CRS VEA model in its multiplier and envelopment form is given as:

$$
\begin{aligned}
\max_{u_j^k, v_i^k} \ & \sum_{j=1}^{J} u_j^k y_j^k \\
s.t. \ & \sum_{j=1}^{J} u_j^k y_j^h - \sum_{i=1}^{I} v_i^k x_i^h \le 0 \quad h = 1, \dots, K, \ h \ne r \\
& \sum_{j=1}^{J} u_j^k y_j^h - \sum_{i=1}^{I} v_i^k x_i^h = 0 \quad h = r = 1, \dots, R \\
& \sum_{i=1}^{I} v_i^k x_i^k = 1 \\
& u_j^k \ge 0 \qquad j = 1, \dots, J \\
& v_i^k \ge 0 \qquad i = 1, \dots, I
\end{aligned}
\qquad
\begin{aligned}
\min_{\theta_{VEA}^k, \lambda_h^k} \ & \theta_{VEA}^k \\
s.t. \ & \sum_{h=1}^{K} \lambda_h^k y_j^h \ge y_j^k \qquad j = 1, \dots, J \\
& \sum_{h=1}^{K} \lambda_h^k x_i^h \le \theta_{VEA}^k x_i^k \quad i = 1, \dots, I \\
& \lambda_h^k \ge 0 \qquad h = 1, \dots, K, \ h \ne r \\
& \lambda_h^k \ free \qquad h = r = 1, \dots, R \\
& \theta_{VEA}^k \ free
\end{aligned}
\tag{5}
$$

where the set R contains the DMUs comprising the MPS. On the other hand, the input-oriented VRS VEA model in its multiplier and envelopment form is given as:

$$
\begin{aligned}
\max_{u_j^k, v_i^k, u^k} \ & \sum_{j=1}^{J} u_j^k y_j^k + u^k \\
s.t. \ & \sum_{j=1}^{J} u_j^k y_j^h - \sum_{i=1}^{I} v_i^k x_i^h + u^k \le 0 \quad h = 1, \dots, K, \ h \ne r \\
& \sum_{j=1}^{J} u_j^k y_j^h - \sum_{i=1}^{I} v_i^k x_i^h + u^k = 0 \quad h = r = 1, \dots, R \\
& \sum_{i=1}^{I} v_i^k x_i^k = 1 \\
& u_j^k \ge 0 \qquad j = 1, \dots, J \\
& v_i^k \ge 0 \qquad i = 1, \dots, I \\
& u^k \ free
\end{aligned}
\qquad
\begin{aligned}
\min_{\theta_{VEA}^k, \lambda_h^k} \ & \theta_{VEA}^k \\
s.t. \ & \sum_{h=1}^{K} \lambda_h^k y_j^h \ge y_j^k \qquad j = 1, \dots, J \\
& \sum_{h=1}^{K} \lambda_h^k x_i^h \le \theta_{VEA}^k x_i^k \quad i = 1, \dots, I \\
& \sum_{h=1}^{K} \lambda_h^k = 1 \\
& \lambda_h^k \ge 0 \qquad h = 1, \dots, K, \ h \ne r \\
& \lambda_h^k \ free \qquad h = r = 1, \dots, R \\
& \theta_{VEA}^k \ free
\end{aligned}
\tag{6}
$$

where the free variable u^k and the convexity constraint for the intensity variables are added in the multiplier and envelopment form, respectively. In the envelopment form of (5) and (6), the non-negativity restrictions are removed from the intensity variables of the DMUs comprising the MPS (Halme *et al.*, 1999). This in turn implies that the inequalities referring to these DMUs should change into strict equalities in the multiplier form of the model, essentially restricting each evaluated DMU to choose input and output weights only among those that are optimal (in the conventional DEA model) for the DMU or the combination of DMUs chosen as the MPS.

3. Main results

3.1. Production trade-offs dual to AR-II type of weight restrictions

To relate the VEA models in (5) and (6) to the DEA models with production-trade-offs and their dual weight restrictions in (2) and (3), notice that each of the side equality restrictions in (5) and (6) can be broken up into the following equivalent pair of inequalities: $\sum_{j=1}^{J} u_j^k y_j^h - \sum_{i=1}^{I} v_i^k x_i^h \leq 0$ and $\sum_{j=1}^{J} u_j^k y_j^h - \sum_{i=1}^{I} v_i^k x_i^h \geq 0$ for (5) and $\sum_{j=1}^{J} u_j^k y_j^h - \sum_{i=1}^{I} v_i^k x_i^h + u^k \leq 0$ and $\sum_{j=1}^{J} u_j^k y_j^h - \sum_{i=1}^{I} v_i^k x_i^h + u^k \geq 0$ for (6). Based on these, (5) and (6) may be rewritten as:

$$
\begin{aligned}
\max_{u_j^k, v_i^k} \ & \sum_{j=1}^{J} u_j^k y_j^k \\
s.t. \ & \sum_{j=1}^{J} u_j^k y_j^h - \sum_{i=1}^{I} v_i^k x_i^h \leq 0 && h = 1, \dots, K \\
& \sum_{j=1}^{J} u_j^k (-y_j^r) - \sum_{i=1}^{I} v_i^k (-x_i^r) \leq 0 && r = 1, \dots, R \\
& \sum_{i=1}^{I} v_i^k x_i^k = 1 \\
& u_j^k \geq 0 && j = 1, \dots, J \\
& v_i^k \geq 0 && i = 1, \dots, I
\end{aligned}
\qquad
\begin{aligned}
\min_{\theta_{VEA}^k, \lambda_h^k} \ & \theta_{VEA}^k \\
s.t. \ & \sum_{h=1}^{K} \lambda_h^k y_j^h + \sum_{r=1}^{R} \pi_r^k (-y_j^r) \geq y_j^k && j = 1, \dots, J \\
& \sum_{h=1}^{K} \lambda_h^k x_i^h + \sum_{r=1}^{R} \pi_r^k (-x_i^r) \leq \theta_{VEA}^k x_i^k && i = 1, \dots, I \\
& \lambda_h^k \geq 0 && h = 1, \dots, K \\
& \pi_r^k \geq 0 && r = 1, \dots, R \\
& \theta_{VEA}^k \ free
\end{aligned}
\tag{7}
$$

and as:

$$
\begin{aligned}
\max_{u_j^k, v_i^k, u^k} \ & \sum_{j=1}^{J} u_j^k y_j^k + u^k \\
s.t. \ & \sum_{j=1}^{J} u_j^k y_j^h - \sum_{i=1}^{I} v_i^k x_i^h + u^k \leq 0 && h = 1, \dots, K \\
& \sum_{j=1}^{J} u_j^k (-y_j^r) - \sum_{i=1}^{I} v_i^k (-x_i^r) - u^k \leq 0 && r = 1, \dots, R \\
& \sum_{i=1}^{I} v_i^k x_i^k = 1 \\
& u_j^k \geq 0 && j = 1, \dots, J \\
& v_i^k \geq 0 && i = 1, \dots, I \\
& u^k \ free
\end{aligned}
\qquad
\begin{aligned}
\min_{\theta_{VEA}^k, \lambda_h^k} \ & \theta_{VEA}^k \\
s.t. \ & \sum_{h=1}^{K} \lambda_h^k y_j^h + \sum_{r=1}^{R} \pi_r^k (-y_j^r) \geq y_j^k && j = 1, \dots, J \\
& \sum_{h=1}^{K} \lambda_h^k x_i^h + \sum_{r=1}^{R} \pi_r^k (-x_i^r) \geq \theta_{VEA}^k x_i^k && i = 1, \dots, I \\
& \sum_{h=1}^{K} \lambda_h^k - \sum_{r=1}^{R} \pi_r^k = 1 \\
& \lambda_h^k \geq 0 && h = 1, \dots, K \\
& \pi_r^k \geq 0 && r = 1, \dots, R \\
& \theta_{VEA}^k \ free
\end{aligned}
\tag{8}
$$

Then one can easily verify that (5) and (7) and (6) and (8) are equivalent to each other.

The additional restrictions in the multiplier form of (7) and (8) result in new terms in the left-hand sides of the inequalities in the envelopment form. These terms contain the negative of the input and output quantities of the DMUs constituting the MPS. We may thus view (7) and (8) as models with $(K+R)$ DMUs, where inputs and outputs take negative values for the DMUs in the set of the MPS (the R additional ones) and positive values for the sample DMUs. By using Emrouznejad *et al.* (2010) data transformations, (7) and (8) may be seen as semi-oriented DEA models with $(K+R)$ DMUs. Specifically, we may redefine the input and output variables in (7) and (8) as:

$$x_{1i}^{h}=\begin{cases}x_i^h, & h=1,\dots,K\\ 0\,, & r=1,\dots,R\end{cases}\quad and\; x_{2i}^{h}=\begin{cases}0, & h=1,\dots,K\\ -x_i^r, & r=1,\dots,R\end{cases}\quad i=1,..,I$$

and:

$$y_{1j}^{h}=\begin{cases}y_j^h, & h=1,\dots,K\\ 0\,, & r=1,\dots,R\end{cases}\quad and\; y_{2j}^{h}=\begin{cases}0, & h=1,\dots,K\\ -y_j^r, & r=1,\dots,R\end{cases}\quad j=1,..,J$$

Then, (7) and (8) are respectively be written as:

$$\begin{aligned}
\max_{u_j^k,v_i^k,u^k}\ & \sum_{j=1}^{J}u_{1j}^{k}y_{1j}^{k}-\sum_{j=1}^{J}u_{2j}^{k}y_{2j}^{k}\\
s.t.\ & \sum_{j=1}^{J}u_{1j}^{k}y_{1j}^{h}-\sum_{j=1}^{J}u_{2j}^{k}y_{2j}^{h}-\\
& -\sum_{i=1}^{I}v_{1i}^{k}x_{1i}^{h}+\sum_{i=1}^{I}v_{2i}^{k}x_{2i}^{h}\le 0 && h=1,\dots,K+R\\
& \sum_{i=1}^{I}v_{1i}^{k}x_{1i}^{h}-\sum_{i=1}^{I}v_{2i}^{k}x_{2i}^{h}=1\\
& u_{1j}^{k}\ge 0 && j=1,\dots,J\\
& u_{2j}^{k}\ge 0 && j=1,\dots,J\\
& v_{2i}^{k}\ge 0 && i=1,\dots,I\\
& v_{2i}^{k}\ge 0 && i=1,\dots,I
\end{aligned}
\qquad
\begin{aligned}
\min_{\theta_{VEA}^k,\lambda_h^k}\ & \theta_{VEA}^{k}\\
s.t.\ & \sum_{h=1}^{K+R}\lambda_h^k y_{1j}^{h}\ge y_{1j}^{k} && j=1,\dots,J\\
& \sum_{h=1}^{K+R}\lambda_h^k y_{2j}^{h}\le y_{2j}^{k} && j=1,\dots,J\\
& \sum_{h=1}^{K+R}\lambda_h^k x_{1i}^{h}\le \theta_{VEA}^{k}x_{1i}^{k} && i=1,\dots,I\\
& \sum_{h=1}^{K+R}\lambda_h^k x_{2i}^{h}\ge \theta_{VEA}^{k}x_{2i}^{k} && i=1,\dots,I\\
& \lambda_h^k\ge 0 && h=1,\dots,K+R\\
& \theta_{VEA}^{k}\ free
\end{aligned}
\qquad (9)$$

and

$$
\begin{aligned}
&\max_{u_j^k, v_i^k, u^k} \sum_{j=1}^{J} u_{1j}^k y_{1j}^k - \sum_{j=1}^{J} u_{2j}^k y_{2j}^k + u^k \\
&s.t. \sum_{j=1}^{J} u_{1j}^k y_{1j}^h - \sum_{j=1}^{J} u_{2j}^k y_{2j}^h - \\
&\quad -\sum_{i=1}^{I} v_{1i}^k x_{1i}^h + \sum_{i=1}^{I} v_{2i}^k x_{2i}^h + u^k \le 0 \qquad h = 1, \dots, K+R \\
&\quad \sum_{i=1}^{I} v_{1i}^k x_{1i}^h - \sum_{i=1}^{I} v_{2i}^k x_{2i}^h = 1 \\
&\quad u_{1j}^k \ge 0 \qquad j = 1, \dots, J \\
&\quad u_{2j}^k \ge 0 \qquad j = 1, \dots, J \\
&\quad v_{2i}^k \ge 0 \qquad i = 1, \dots, I \\
&\quad v_{2i}^k \ge 0 \qquad i = 1, \dots, I \\
&\quad u^k \; free
\end{aligned}
\qquad
\begin{aligned}
&\min_{\theta_{VEA}^k, \lambda_h^k} \theta_{VEA}^k \\
&s.t. \sum_{h=1}^{K+R} \lambda_h^k y_{1j}^h \ge y_{1j}^k \qquad j = 1, \dots, J \\
&\quad \sum_{h=1}^{K+R} \lambda_h^k y_{2j}^h \le y_{2j}^k \qquad j = 1, \dots, J \\
&\quad \sum_{h=1}^{K+R} \lambda_h^k x_{1i}^h \le \theta_{VEA}^k x_{1i}^k \quad i = 1, \dots, I \\
&\quad \sum_{h=1}^{K+R} \lambda_h^k x_{2i}^h \ge \theta_{VEA}^k x_{2i}^k \quad i = 1, \dots, I \\
&\quad \sum_{h=1}^{K+R} \lambda_h^k = 1 \\
&\quad \lambda_h^k \ge 0 \qquad h = 1, \dots, K+R \\
&\quad \theta_{VEA}^k \; free
\end{aligned}
\tag{10}
$$

where h is used to index all DMUs, i.e., $h=1,\dots,K+R$.

We can now provide sufficient conditions under which the DEA model including production trade-offs or their dual weight restrictions is equivalent to the VEA model. Under CRS, a comparison of (2) and (7) shows that the two models are equivalent to each other if the number of trade-offs in the former is equal to the number of DMUs constituting the MPS in the latter and the trade-off coefficient vectors are given as:

$$
\begin{aligned}
P^r &= [-x_1^r, \dots, -x_i^r, \dots, -x_I^r]^T, \;\; r = 1, \dots, R \\
Q^r &= \left[-y_1^r, \dots, -y_j^r, \dots, -y_J^r\right]^T, \;\; r = 1, \dots, R
\end{aligned}
\tag{11}
$$

where $\left(x_i^r, y_j^r\right)$ correspond to the inputs and outputs of each of the DMUs ($r=1,\dots,R$) constituting the MPS. The trade-offs in (11) are dual to the following set of AR-II type weight restrictions:

$$
\sum_{j=1}^{J} u_j^k\left(-y_j^r\right) - \sum_{i=1}^{I} v_i^k(-x_i^r) \le 0, \;\; r = 1, \dots, R \tag{12}
$$

which are essentially the same as the second set of restrictions in the multiplier form of (7). Thus, we have:

PROPOSITION 1: Under constant returns to scale, the VEA model is equivalent to a DEA model including production trade-offs, for which the trade-off coefficient vectors contain the negative of the input and output quantities of the DMUs constituting the MPS in VEA.

When VRS is assumed, substituting (11) or its dual (12) into (3) will not result in a model equivalent to (8), since the convexity constraints in the envelopment form of (3) and (8) are different from each other.

However, we can show that the VEA model is related to the DEA model including another form of trade-offs:

PROPOSITION 2: Regardless of the nature of the returns to scale, the VEA model is equivalent to a DEA model including production trade-offs, for which the trade-off coefficient vectors contain the deviations of each DMU's input and output quantities from those of each of the DMUs constituting the MPS.

To show this, consider the following trade-offs:

$$\begin{aligned} P_r^h &= \left[\left(x_1^h - x_1^r\right), \ldots, \left(x_I^h - x_I^r\right)\right]^T, \;\; h = 1, \ldots, K, \;\; r = 1, \ldots, R \\ Q_r^h &= \left[\left(y_1^h - y_1^r\right), \ldots, \left(y_J^h - y_J^r\right)\right]^T, \;\; h = 1, \ldots, K, \;\; r = 1, \ldots, R \end{aligned} \tag{13}$$

which are dual to the following set of AR-II type of weight restrictions:

$$\sum_{j=1}^{J} u_j^k\left(y_j^h - y_j^r\right) - \sum_{i=1}^{I} v_i^k\left(x_i^h - x_i^r\right) \leq 0, \;\; h = 1, \ldots, K, \;\; r = 1, \ldots, R \tag{14}$$

Let's assume, initially, that $r = 1$, i.e., the MPS is a single DMU. Then, (13) consists of K trade-off coefficient vectors given as the deviations of each DMU's *(h=1,…,K)* input and output quantities of from those of the MPS. That is, $p_i^h = \left(x_i^h - x_i^r\right),\; i = 1, \ldots, I,\; h = 1, \ldots, K$ and $q_j^h = \left(y_j^h - y_j^r\right),\; j = 1, \ldots, J,\; h = 1, \ldots, K$. In such a case, the envelopment form of the VRS DEA model in (3) is given as:

$$
\begin{aligned}
&\min_{\theta_{TO}^k,\lambda_h^k,\pi_h^k} \quad \theta_{TO}^k \\
&s.t. \quad \sum_{h=1}^{K}\lambda_h^k y_j^h + \sum_{h=1}^{K}\pi_h^k\left(y_j^h - y_j^r\right) \geq y_j^k \qquad j = 1,\dots,J \\
&\qquad \sum_{h=1}^{K}\lambda_h^k x_i^h + \sum_{h=1}^{K}\pi_h^k\left(x_i^h - x_i^r\right) \leq \theta_{TO}^k x_i^k \quad i = 1,\dots,I \\
&\qquad \sum_{h=1}^{K}\lambda_h^k = 1 \\
&\qquad \lambda_h^k \geq 0 \qquad\qquad h = 1,\dots,K \\
&\qquad \pi_h^k \geq 0 \qquad\qquad h = 1,\dots,K \\
&\qquad \theta_{TO}^k \quad free
\end{aligned}
\tag{15}
$$

or equivalently as:

$$
\begin{aligned}
&\min_{\theta_{TO}^k,\delta_h^k,\gamma^k} \quad \theta_{TO}^k \\
&s.t. \quad \sum_{h=1}^{K}\delta_h^k y_j^h + \gamma^k\left(-y_j^r\right) \geq y_j^k \qquad j = 1,\dots,J \\
&\qquad \sum_{h=1}^{K}\delta_h^k x_i^h + \gamma^k(-x_i^r) \leq \theta_{TO}^k x_i^k \quad i = 1,\dots,I \\
&\qquad \sum_{h=1}^{K}\delta_h^k - \gamma^k = 1 \\
&\qquad \delta_h^k \geq 0 \qquad\qquad h = 1,\dots,K \\
&\qquad \gamma^k \geq 0 \\
&\qquad \theta_{TO}^k \quad free
\end{aligned}
\tag{16}
$$

where $\delta_h^k = \left(\lambda_h^k + \pi_h^k\right) \geq 0$ and $\sum_{h=1}^{K}\pi_h^k = \gamma^k \geq 0$. Then (16) is equivalent to the envelopment form in (8) if the *r*th DMU is chosen as the MPS. If $r > 1$, namely that the MPS is a combination of several DMUs, then (13) consists of $K \times R$ trade-off coefficient vectors given as the deviations of each DMU's *(h=1,…,K)* input and output quantities from those of each DMU *(r=1,…,R)* comprising the MPS. As a result, the second term in the left hand side of the first two inequality restrictions in (15) reflect summations over both *h* (*h=1,…,K)* and *r (r=1,…,R)* and π_h^k should be changed to π_{hr}^k. Moreover, by defining $\delta_h^k = \left(\lambda_h^k + \sum_{r=1}^{R}\pi_{hr}^k\right) \geq 0$ and $\gamma_r^k = \sum_{h=1}^{K}\pi_h^k \geq 0$ we may obtain a model similar to (16) in which the second term in the left hand side of the first two inequality restrictions reflect summations over *r (r=1,…,R)* and the third restriction is stated as $\sum_{h=1}^{K}\delta_h^k - \sum_{r=1}^{R}\gamma_r^k = 1$. This model is equivalent to the envelopment form in (8) if the set of *R (r=1,…,R)* DMUs comprise the MPS. In a similar fashion, if increasing (decreasing) returns to scale are assumed, then the equality sign in the third restriction of the envelopment forms in (3) and (8) and in (15) and (16) is simply changed to a less-than-or-equal (greater-than-or-equal) sign, while if constant returns to scale are assumed, the third restriction in the envelopment forms in (3) and (8) and in (15) and (16) should be dropped.

From the above, it is also clear that, under CRS, the DEA model with *R* trade-off coefficient vectors, given as the negative of the input and output quantities of the DMUs selected as the MPS in VEA, is equivalent to the DEA model with $(K \times R)$ trade-off coefficient vectors, given as the deviations of each DMU's (*h=1,…,K*) input and output quantities from those of each of the DMUs selected as the MPS in VEA. This is evident as long as the trade-off coefficient vectors in (2) are given as in either (11) or (13). Let's assume, initially, that $r = 1$, i.e., the MPS is a single DMU. Then, the envelopment form of (2) when the trade-off coefficient vectors are given by (13) is:

$$\begin{aligned}
&\min_{\theta_{TO}^k,\lambda_h^k,\pi_h^k} \ \theta_{TO}^k \\
&s.t.\ \sum_{h=1}^{K}\lambda_h^k y_j^h + \sum_{h=1}^{K}\pi_h^k\left(y_j^h - y_j^r\right) \geq y_j^k \qquad j = 1,\dots,J \\
&\quad\ \sum_{h=1}^{K}\lambda_h^k x_i^h + \sum_{h=1}^{K}\pi_h^k\left(x_i^h - x_i^r\right) \leq \theta_{TO}^k x_i^k \quad i = 1,\dots,I \\
&\quad\ \lambda_h^k \geq 0 \qquad h = 1,\dots,K \\
&\quad\ \pi_h^k \geq 0 \qquad h = 1,\dots,K \\
&\quad\ \theta_{TO}^k \quad free
\end{aligned} \tag{18}$$

while when the trade-off coefficient vectors are given by (11), it is as follows:

$$\begin{aligned}
&\min_{\theta_{TO}^k,\zeta_h^k,\gamma^k} \ \theta_{TO}^k \\
&s.t.\ \sum_{h=1}^{K}\zeta_h^k y_j^h + \beta^k\left(-y_j^r\right) \geq y_j^k \qquad j = 1,\dots,J \\
&\quad\ \sum_{h=1}^{K}\zeta_h^k x_i^h + \beta^k(-x_i^r) \leq \theta_{TO}^k x_i^k \quad i = 1,\dots,I \\
&\quad\ \zeta_h^k \geq 0 \qquad h = 1,\dots,K \\
&\quad\ \beta^k \geq 0 \\
&\quad\ \theta_{TO}^k \quad free
\end{aligned} \tag{19}$$

If $\zeta_h^k = \left(\lambda_h^k + \pi_h^k\right) \geq 0$ and $\sum_{h=1}^{K}\pi_h^k = \beta^k \geq 0$, then (18) is equivalent to (19). If $r > 1$, then (13) consists of $(K \times R)$ trade-off coefficient vectors given as $p_i^{hr} =$

$\left(x_i^h - x_i^r\right),\ i = 1,\dots,I,\ h = 1,\dots,K, r = 1,\dots,R$ and $q_j^{hr} = \left(y_j^h - y_j^r\right),\ j = 1,\dots,J,\ h = 1,\dots,K, r = 1,\dots,R$. Thus, the second terms in the left hand side of the first two inequality restrictions in (18) reflect summations over both h (*h=1,…,K)* and r *(r=1,…,R)* and π_h^k should be changed to π_{hr}^k. Furthermore, (11) consists of R trade-off coefficient vectors given by the negative of the input and output quantities of the DMUs chosen as the MPS in VEA. Thus, the second terms in the left hand side of the first two inequality restrictions in (19) reflect summations over r *(r=1,…,R)* and β^k should be changed to β_r^k. Then, by defining $\zeta_h^k = \left(\lambda_h^k + \sum_{r=1}^{R} \pi_{hr}^k\right) \geq 0$ and $\beta_r^k = \sum_{h=1}^{K} \pi_h^k \geq 0$, (18) is equivalent to (19). Consequently, a CRS DEA model augmented with the trade-offs as in (11) and a CRS DEA model augmented with the trade-offs as in (13) are equivalent to each other.

The above results indicate that, under certain circumstances, the DM preferences underlying the evaluation of DMUs in the VEA model may be seen as a particular form of trade-offs or AR-II type of weight restrictions and vice versa. This provides an alternative interpretation of the efficiency scores obtained from both the VEA model and the DEA model including production trade-offs.

The production trade-offs in (11) and (13) and their dual weight restrictions in (12) and (14) enlarge the DEA efficient frontier by extending certain of its existing facets, in particular, those associated with the DMU or the combination of DMUs comprising the MPS, instead of introducing new linear segments.[6] The implications of this are: (i) (11) and (13) do not introduce additional information in the envelopment form of the DEA model other than that already implicit in the data, namely the rates of substitution between inputs, the rates of transformation among outputs, and the marginal products between inputs and outputs that are reflected in each of the extended facets, and (ii) the efficiency scores from the multiplier form of the DEA models with (12) and (14) do not underestimate the true efficiency of the evaluated DMUs, as may occur in several other cases where additional restrictions of the general form in (4) are imposed in DEA models (see Tracy and Chen, 2005; Khalili *et al.*, 2010b). This is because each facet of the DEA frontier enlarged with (12) or (14) is already tangent to the conventional DEA frontier at some point.

3.2. Production trade-offs dual to AR-I type of weight restrictions

In the previous section we considered production trade-offs related to both inputs and outputs, which are dual to AR-II type of weight restrictions. In this section we consider weight restrictions of the AR-I type, and we restrict our attention to pure input or output models, i.e., models that contain respectively no outputs and no inputs.

Consider first the DEA model without inputs, which is equivalent to a DEA model with a single or multiple constant (unitary) inputs (Lovell and Pastor, 1999).[7] The latter is known as the Benefit-of-the-Doubt model (BoD) and its multiplier and envelopment form are given as (Cherchye *et al.*, 2007):

$$\begin{aligned} &\max_{u_j^k, v_i^k} \sum_{j=1}^{J} u_j^k y_j^k \\ &s.t. \ \sum_{j=1}^{J} u_j^k y_j^h \le 1 \quad h = 1, \dots, K \\ &\quad u_j^k \ge 0 \qquad j = 1, \dots, J \end{aligned} \qquad \begin{aligned} &\min_{\lambda_h^k} \sum_{h=1}^{K} \lambda_h^k \\ &s.t. \ \sum_{h=1}^{K} \lambda_h^k y_j^h \ge y_j^k \quad j = 1, \dots, J \\ &\quad \lambda_h^k \ge 0 \qquad h = 1, \dots, K \\ &\quad \theta^k \ free \end{aligned} \tag{20}$$

The model in (20) is obtained from (2) by dropping the terms associated with the (input and output) trade-offs or their dual weight restrictions, and by considering that $i = 1,\ x^h = 1, h = 1, \dots, K$ which implies that $v^k = 1$.[8] The BoD model has recently been adapted to a VEA framework (see Ravanos and Karagiannis, 2021) and its multiplier and envelopment form are given as:

$$\begin{aligned} &\max_{u_j^k} \sum_{j=1}^{J} u_j^k y_j^k \\ &s.t. \ \sum_{j=1}^{J} u_j^k y_j^h \le 1 \quad h = 1, \dots, K,\ h \ne r \\ &\quad \sum_{j=1}^{J} u_j^k y_j^h = 1 \quad h = r = 1, \dots, R \\ &\quad u_j^k \ge 0 \qquad j = 1, \dots, J \end{aligned} \qquad \begin{aligned} &\min_{\lambda_h^k} \sum_{h=1}^{K} \lambda_h^k \\ &s.t. \ \sum_{h=1}^{K} \lambda_h^k y_j^h \ge y_j^k \quad j = 1, \dots, J \\ &\quad \lambda_h^k \ge 0 \qquad h = 1, \dots, K,\ h \ne r \\ &\quad \lambda_h^k \ free \qquad h = r = 1, \dots, R \end{aligned} \tag{21}$$

where the set R contains the DMUs comprising the MPS.

For $i = 1$ and $x^h = 1, h = 1, \dots, K, \left(x^h - x^r\right) = 0, h = 1, \dots, K, r = 1, \dots, R,$ and thus the vector P_r^h in (13) is a scalar with a value equal to zero. Consequently, the associated weight restrictions in (14) consider only outputs, i.e., are of the AR-I type, as the second component in each of the relations in (14) is equal to zero. In a similar fashion, the vector P_r in (11) is a scalar with a value equal to -1 and the second component in each of the associated weight restrictions in (12) is also equal to -1, namely (12) considers only outputs. Thus, we can show the following:

PROPOSITION 3: The VEA BoD model is equivalent to a BoD model including production trade-offs, for which the trade-off coefficient vectors contain either (i) the negative of the output quantities of the DMUs constituting the MPS, or (ii) the deviations of each DMU's output quantities from those of each of the DMUs constituting the MPS.

Next, consider the DEA model without outputs, which is equivalent to a DEA model with a single or multiple constant (unitary) outputs (Lovell and Pastor, 1999).[9] The latter is known as the Inverted BoD model and its multiplier and envelopment form are given as (Färe and Karagiannis, 2014): [10]

$$
\begin{array}{llll}
\min\limits_{v_i^k} \sum\limits_{i=1}^{I} v_i^k x_i^k & & \max\limits_{\lambda_h^k} \sum\limits_{h=1}^{K} \lambda_h^k & \\
s.t. \ \sum\limits_{i=1}^{I} v_i^k x_i^h \geq 1 & h = 1,\dots,K & s.t. \ \sum\limits_{h=1}^{K} \lambda_h^k x_i^h \leq x_i^k & i = 1,\dots,I \\
\quad v_i^k \geq 0 & i = 1,\dots,I & \quad \lambda_h^k \geq 0 & h = 1,\dots,K
\end{array}
\tag{22}
$$

The model in (22) is obtained from the output-oriented counterpart of (2) by dropping the terms associated with the (input and output) trade-offs or their dual weight restrictions, and by assuming that $j = 1,\ y^h = 1, h = 1, \dots, K$, which implies that $u^k = 1$. Compared to the BoD model, the Inverted BoD model provides a pessimistic perspective of performance evaluation (Karagiannis, 2021). Consider now a set of *R* DMUs reflecting the most desirable input bundle from DM's point of view. Then, the multiplier and envelopment form of the Inverted VEA BoD model will be given as:

$$
\begin{array}{llll}
\min\limits_{v_i^k} & \sum\limits_{i=1}^{I} v_i^k x_i^k & & \\
s.t. & \sum\limits_{i=1}^{I} v_i^k x_i^h \geq 1 & h = 1, \dots, K, \;\; h \neq r & \\
& \sum\limits_{i=1}^{I} v_i^k x_i^h = 1 & h = r = 1, \dots, R & \\
& v_i^k \geq 0 & i = 1, \dots, I &
\end{array}
\qquad
\begin{array}{lll}
\max\limits_{\lambda_h^k} & \sum\limits_{h=1}^{K} \lambda_h^k & \\
s.t. & \sum\limits_{h=1}^{K} \lambda_h^k x_i^h \leq x_i^k & i = 1, \dots, I \\
& \lambda_h^k \geq 0 & h = 1, \dots, K, \;\; h \neq r \\
& \lambda_h^k \; free & h = r = 1, \dots, R
\end{array}
\tag{23}
$$

Then for $j = 1$ and $y^h = 1, h = 1, \dots, K,$ we have that $(y^h - y^r) = 0, h = 1, \dots, K, r = 1, \dots, R$ and thus the vector Q_r^h in (13) is a scalar than takes the value of zero. Thus, the weights restrictions dual to the production trade-offs in (13) are AR-I, as the first component in each of the relations in (14) is equal to zero. Similarly, the vector Q_r in (11) is a scalar with a value equal to -1 and the same holds for the first component in each of the associated weight restrictions in (12). Thus, we can show that:

PROPOSITION 4: The Inverted BoD VEA model is equivalent to an Inverted BoD model including production trade-offs, for which the trade-off coefficient vectors contain either (i) the negative of the input quantities of the DMUs constituting the MPS, or (ii) the deviations of each DMU's input quantities from those of each of the DMUs constituting the MPS.

4. An empirical application

To illustrate the usefulness of our findings, we consider the case of a DM evaluating alternatives in a technology selection problem, using the dataset of 27 industrial robots in Khouja (1995) and Baker and Talluri (1997).[11] For the purposes of the present paper, we may consider the DM assessing these 27 DMUs as either a potential buyer, i.e., the manager of an industrial plant, or a technology manufacturer, namely the owner of a company producing one of the assessed DMUs.

Data for the 27 DMUs are given in columns 2 to 5 of Table 1. Four among the most important performance features of industrial robots are considered, which are (i) the robots' cost (in 10.000$), (ii) repeatability, namely a measure of the distance (in mm) covered by the robot in repeated trials, (iii) the robot's payload capacity (in kg) and (iv) its minimum possible velocity (in m/s). For the former two features lower values indicate better performance, and hence they are treated as inputs, while larger values are more preferable for capacity and velocity and these are treated as outputs (Khouja, 1995). Efficiency estimates based on the input-oriented CRS and VRS DEA models are given in columns 6 and 9 of Table 1. From these, we can see that nine DMUs are efficient with CRS, while other two DMUs are added to the list of efficient DMUs in the VRS model. The assumption of VRS results in a noticeable increase in average efficiency (0.801 compared to 0.725 in the CRS model).

For our purposes let's assume that DMU #19 is chosen as the MPS. The DM in this case may be the manager of a manufacturing plant that operates using this particular robot, or be a potential buyer for which this robot has an attractive combination of low cost and low repeatability. The CRS and VRS VEA efficiency scores when DMU #19 is chosen as the MPS are given in columns 7 and 10 of Table 1. In the CRS case, five DMUs drop from the list of efficient DMUs compared to DEA, while when VRS is assumed the efficient DMUs are reduced to eight, compared to 11 in DEA. By Proposition 2, the same efficiency scores would result from respectively a CRS and a VRS DEA model including the following set of weight restrictions:

$$
\begin{array}{rrrrll}
50.000u_1^k & +1.050u_2^k & -6.260v_1^k & -0.100v_2^k & \leq 0 & \text{(DMU \#1)} \\
-4.000u_1^k & +0.800u_2^k & -3.860v_1^k & & \leq 0 & \text{(DMU \#2)} \\
35.000u_1^k & +0.970u_2^k & -4.060v_1^k & -1.220v_2^k & \leq 0 & \text{(DMU \#3)} \\
-8.500u_1^k & +0.360u_2^k & -6.260v_1^k & +0.025v_2^k & \leq 0 & \text{(DMU \#4)} \\
40.000u_1^k & -0.250u_2^k & -8.660v_1^k & -0.200v_2^k & \leq 0 & \text{(DMU \#5)} \\
-9.000u_1^k & & -0.130v_1^k & -0.050v_2^k & \leq 0 & \text{(DMU \#6)} \\
-5.000u_1^k & +0.700u_2^k & -0.820v_1^k & -0.050v_2^k & \leq 0 & \text{(DMU \#7)} \\
5.000u_1^k & +0.700u_2^k & -2.260v_1^k & -0.050v_2^k & \leq 0 & \text{(DMU \#8)} \\
 & +0.800u_2^k & -5.780v_1^k & -0.150v_2^k & \leq 0 & \text{(DMU \#9)} \\
-4.000u_1^k & +0.700u_2^k & -1.460v_1^k & & \leq 0 & \text{(DMU \#10)} \\
20.000u_1^k & +0.600u_2^k & -1.940v_1^k & -0.450v_2^k & \leq 0 & \text{(DMU \#11)} \\
3.600u_1^k & -0.150u_2^k & -5.960v_1^k & -0.950v_2^k & \leq 0 & \text{(DMU \#12)} \\
 & 0.900u_2^k & -2.260v_1^k & & \leq 0 & \text{(DMU \#13)} \\
20.000u_1^k & +0.900u_2^k & -3.060v_1^k & & \leq 0 & \text{(DMU \#14)} \\
37.000u_1^k & +0.700u_2^k & -2.740v_1^k & -0.950v_2^k & \leq 0 & \text{(DMU \#15)} \\
70.000u_1^k & +0.700u_2^k & -5.940v_1^k & -0.950v_2^k & \leq 0 & \text{(DMU \#16)} \\
5.000u_1^k & +1.700u_2^k & -7.060v_1^k & -1.950v_2^k & \leq 0 & \text{(DMU \#17)} \\
 & 0.700u_2^k & -5.360v_1^k & -0.150v_2^k & \leq 0 & \text{(DMU \#18)} \\
-8.500u_1^k & +0.500u_2^k & +0.780v_1^k & -1.950v_2^k & \leq 0 & \text{(DMU \#20)} \\
17.000u_1^k & +1.400u_2^k & -1.870v_1^k & -1.950v_2^k & \leq 0 & \text{(DMU \#21)} \\
-9.100u_1^k & +0.700u_2^k & -2.860v_1^k & & \leq 0 & \text{(DMU \#22)} \\
-7.500u_1^k & +0.200u_2^k & -0.310v_1^k & -0.050v_2^k & \leq 0 & \text{(DMU \#23)} \\
-7.500u_1^k & +0.200u_2^k & -0.430v_1^k & -0.050v_2^k & \leq 0 & \text{(DMU \#24)} \\
 & 0.700u_2^k & -2.690v_1^k & -0.150v_2^k & \leq 0 & \text{(DMU \#25)} \\
60.000u_1^k & +0.950u_2^k & -4.360v_1^k & -1.220v_2^k & \leq 0 & \text{(DMU \#26)} \\
195.000u_1^k & +0.450u_2^k & -3.060v_1^k & -1.980v_2^k & \leq 0 & \text{(DMU \#27)}
\end{array}
$$

The figures attached to the input and output weights in each of the above restrictions are equal to the deviations of the input and output quantities corresponding to the evaluated DMU listed in parentheses, from those of DMU #19. This set of restrictions forces the marginal rates of substitution and transformation for the evaluated DMUs to take values only within the range of marginal rates prevailing on the efficient frontier in the neighborhood of DMU #19. Note that by Proposition 1 the CRS VEA efficiency scores could also be obtained through a DEA model including the following weight restriction:

$$
-10.000u_1^k \quad -0.300u_2^k \quad +0.940v_1^k \quad +0.050v_2^k \quad \leq 0
$$

the coefficients of which are the negative of the input and output quantities of DMU #19.

Let us now assume that the following trade-off is included in the envelopment form of the CRS DEA model in (1):

$$P = [-0.160, -2.000]^T$$
$$Q = [-1.500, -0.800]^T$$

This trade-off implies that, if the DM is willing to accept a decrease in load capacity by 1.500 kg and in velocity by 0.800 m/s, then the robot's cost and repeatability could be decreased by at most 1600$ and 2.000 mm respectively. The above trade-off coefficient vectors are equal to the negative of the input and output quantities of the DEA-efficient DMU #20. Thus, by Proposition 1, a CRS DEA model augmented with this trade-off is equivalent to a CRS VEA model in which DMU #20 is chosen as the MPS. The resulting efficiency scores when either the above trade-off is included in the CRS DEA model, or DMU #20 is the MPS in the CRS VEA model are given in column 8 of Table 1. Compared to the DEA results, we see that only three DMUs remain efficient in the VEA model, while average efficiency decreases to 0563.

The efficiency scores from a VRS VEA model in which DMU #20 is used as the MPS are given in column 11 of Table 1. By Proposition 2, these scores can be obtained via a DEA model including production trade-offs the coefficient vectors of which contain the deviations of each DMU's input and output quantities from those of DMU #20.

5. Concluding remarks

In this paper, we have examined the links between DEA models with weights restrictions or their dual production trade-offs and VEA and we showed that VEA may be viewed as a class of DEA models with particular trade-offs. More specifically, we showed that, irrespective of the nature of the returns to scale, the VEA model is equivalent to the DEA model including production trade-offs, for which the trade-off coefficient vectors are given by the deviations of the input and output quantities of each sample DMU from those of DMUs chosen as the MPS. In addition, with CRS, the VEA model is equivalent to the DEA model with trade-off coefficient vectors given by the negative of the input and output quantities of the DMUs chosen as the MPS in VEA. These trade-offs are dual to AR-II weight restrictions. In addition, we showed that, when we are considering these particular trade-offs only for the inputs or the outputs, a similar equivalence results between the pure output or input VEA models and their DEA counterparts including trade-offs. In these cases, the dual forms of the trade-offs are AR-I weight restrictions.

The results in this paper indicate that the DM preferences about the most preferred input/output bundle as reflected in the MPS in the VEA model may be seen as a particular form of trade-offs or their dual AR-II or AR-I type of weight restrictions and vice versa. This provides an alternative interpretation of the efficiency scores obtained from both the VEA model and its equivalent DEA model including production trade-offs. In particular, the VEA efficiency scores can also be interpreted as including restrictions in the acceptable values of the marginal rates of substitution and transformation, while it may be said that the efficiency scores obtained from the DEA model including production trade-offs reflect the DM's judgements about the most preferred input/output bundle. Promising avenues for future research would be to investigate the potential relationships between VEA and other forms of introducing restrictions in DEA models, such as cone-ratio DEA, as well as other types of performance evaluation models that take preferences into account, such as cross efficiency formulations.

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Table 1: Data and efficiency estimates for the illustrative example.

DMU	inputs		outputs		CRS models			VRS models		
	Cost (in 10.000$)	Repeatability (in mm)	Load capacity (in kg)	Velocity (in m/s)	DEA	VEA (MPS: #19)	VEA (MPS: #20)	DEA	VEA (MPS: #19)	VEA (MPS: #20)
1	7.200	0.150	60.000	1.350	1.000	0.871	0.479	1.000	1.000	0.512
2	4.800	0.050	6.000	1.100	0.904	0.511	0.417	0.907	0.868	0.437
3	5.000	1.270	45.000	1.270	0.529	0.465	0.529	0.667	0.507	0.568
4	7.200	0.030	1.500	0.660	1.000	0.195	0.167	1.000	1.000	0.196
5	9.600	0.250	50.000	0.050	0.592	0.392	0.108	0.594	0.594	0.177
6	1.070	0.100	1.000	0.300	0.482	0.414	0.482	0.865	0.865	0.865
7	1.760	0.100	5.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
8	3.200	0.100	15.000	1.000	0.783	0.783	0.618	0.783	0.783	0.618
9	6.720	0.200	10.000	1.100	0.378	0.362	0.306	0.383	0.376	0.315
10	2.400	0.050	6.000	1.000	1.000	0.891	0.758	1.000	1.000	0.759
11	2.880	0.500	30.000	0.900	0.671	0.671	0.669	0.677	0.677	0.677
12	6.900	1.000	13.600	0.150	0.102	0.099	0.069	0.142	0.142	0.142
13	3.200	0.050	10.000	1.200	1.000	0.874	0.701	1.000	0.990	0.744
14	4.000	0.050	30.000	1.200	1.000	1.000	0.658	1.000	1.000	0.691
15	3.680	1.000	47.000	1.000	0.613	0.561	0.613	0.624	0.607	0.623
16	6.880	1.000	80.000	1.000	0.604	0.592	0.437	0.604	0.604	0.441
17	8.000	2.000	15.000	2.000	0.405	0.272	0.405	1.000	0.362	0.525
18	6.300	0.200	10.000	1.000	0.365	0.355	0.299	0.367	0.367	0.299
19	0.940	0.050	10.000	0.300	1.000	1.000	0.733	1.000	1.000	1.000
20	0.160	2.000	1.500	0.800	1.000	0.169	1.000	1.000	1.000	1.000
21	2.810	2.000	27.000	1.700	0.852	0.397	0.852	1.000	0.774	1.000
22	3.800	0.050	0.900	1.000	0.829	0.509	0.476	0.913	0.906	0.477
23	1.250	0.100	2.500	0.500	0.694	0.648	0.694	0.923	0.923	0.923
24	1.370	0.100	2.500	0.500	0.636	0.606	0.636	0.847	0.847	0.847
25	3.630	0.200	10.000	1.000	0.553	0.553	0.511	0.556	0.556	0.511
26	5.300	1.270	70.000	1.250	0.581	0.581	0.577	0.771	0.582	0.613
27	4.000	2.030	205.000	0.750	1.000	1.000	1.000	1.000	1.000	1.000
average	4.224	0.589	28.315	0.929	0.725	0.584	0.563	0.801	0.753	0.628

Note: The data are taken from Baker and Talluri (1997).

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Endnotes

[1] This is also one of the main reasons motivating the incorporation of weights restrictions

[2] We focus on the input-oriented model, but our results can be straightforwardly extended to the output-oriented model.

[3] Podinovski (2004) has shown that non-homogeneous linear weight restrictions, i.e., those for which the right-hand side of (4) is non-zero, can also be represented in the form of production trade-offs in the envelopment form of the DEA model.

[4] Cases (i) and (ii) may also refer to a subset of inputs and outputs if respectively $p_i^r = 0$ for some *i* in Case (i) and $q_j^r = 0$ for some *j* in Case (ii).

[5] In Charnes *et al.* (1991) the DMUs with a DEA efficiency score of one are classified into three categories: (i) extreme-efficient DMUs that reside at a point of the convex DEA frontier where more than one facets intercept, (ii) non-extreme-efficient DMUs, namely DMUs located on the interior of a facet, and (iii) weakly-efficient DMUs that have at least one positive optimal value for an input or output slack. An MPS chosen by the DMUs in (ii) can be expressed as a linear combination of DMUs in (i). If the DM chooses a dominated (i.e., DEA-inefficient or weakly-efficient) DMU as the MPS, then his/her DM preferences can be stated equivalently by using as the MPS the combination of the extreme-efficient DMUs that are identified as peers of the chosen DMU, as the MPS (see Halme *et al.,* 1999).

[6] Weight restrictions that result in extending facets of the DEA frontier are discussed in Portela and Thanassoulis (2006), but are not related to VEA.

[7] Note that when we consider only outputs it makes no sense to have an input-oriented model. Also, as Lovell and Pastor (1999) have shown, a pure-output CRS output-oriented DEA model rates all DMUs as infinitely inefficient, while an input-oriented VRS DEA model with a single constant input rates all DMUs as efficient.

[8] Variants of (20) including weight restrictions have been employed in, among others, the construction of composite indicators of environmental performance (Zanella *et al.*, 2013), the re-estimation of the Technology Achievement Index (Cherchye *et al.*, 2008), and the aggregation of several measures of money into a synthetic indicator (Sahoo and Acharya, 2010).

[9] Note that when we consider only inputs it makes no sense to have an output-oriented model. Also, as Lovell and Pastor (1999) have shown, a pure-input CRS input-

oriented DEA model rates all DMUs as infinitely inefficient, while an output-oriented VRS DEA model with a single constant output rates all DMUs as efficient.

[10] Variants of (22) including weight restrictions have been used by, among others, Zhou *et al.* (2007) to construct a sustainable energy index, and Rogge (2012) to re-estimate the Environmental Performance Index.

[11] The applications of DEA and other multi-criteria decision-making methods in technology selection are nowadays voluminous and include, but are not limited to, the selection of flexible manufacturing systems, industrial robots, and dispatching rules. A review of such applications is a task out of the scope of the present paper, and the interested reader is referred to Hamzeh and Xu (2019), for a recent review.