A Model of Economic Growth and Long Cycles

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The purpose of this article is to derive an endogenous growth and cycles model that integrates the general law of capital accumulation, the reserve army of labor, technological change, devaluation of capital, and the law of the tendential fall in the rate of profit. The phase space of this model is analyzed by estimating its equilibrium solutions and exploring its economically meaningful stability properties. The so derived endogenous growth cum cycles model is then simulated with realistic parameter values. The solutions of our model display periodicity, which may be treated as long cycles like attractors conditioned by the long–run movement of profitability. The salient feature of our model is the growth of the rate of surplus value, which becomes the regulating variable of our system for it explains both the deviations from equilibrium and at the same time provides a realistic solution to Harrodian instability.

***Keywords*:** capital accumulation, falling profitability, technological change, devaluation of capital, reserve army of labor, economic growth, long cycles

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1. Introduction

The Harrod–Domar and Solow–Swan growth models took for granted that saving, investment, and technological change are exogenously determined variables. In particular, the first growth models assumed either that investment in combination with capacity utilization gives rise to optimal economic growth (Harrod 1939; Domar 1948) or a given rate of technological change such that to determine the level of utilized labor and capital (Solow 1957, 1960; Phelps 1961). Since then, there have been many models that endogenize saving, investment, and technological progress, however with little or no success, as this can be judged by the number of issues raised among contending narratives echoing the diverse conceptualizations of the variables involved.

More specifically, neoclassical and neo–Keynesian approaches attempt to endogenize savings through either households' preferences, firms' choices with respect to alternative production techniques, or even government policies. However, as Darity (2009) argues, the introduction of these features in a Solow–like growth model may give rise to instability and chaotic behavior. Moreover, mainstream economists, in their effort to explain Solow's residual, introduce technological change and investment behavior by bringing in their analysis features such as R&D expenditures, human capital, and shocks of innovation. However, in doing so, they fail to explain both, theoretically and empirically, the widening gap in the development between countries let alone the modeling of economic growth and long cycles (Parente 2001). The post–Keynesian approaches, in modelling economic growth, emphasized the effects of profitability on investment, productivity, and capacity utilization, while Robinson (1965) discussed the effects of innovations on labor productivity. Kaldor (1957), on the other hand, examined how technological change affects the "productivity" of capital. Furthermore, a central issue in the growth literature is

Harrod's "knife–edge" solution; the efforts to resolve it ended up within a comparative static framework parting with the dynamic features of growth and real time (Bellais 2004).

The modeling of economic growth and cycles continued within the static general equilibrium framework; by contrast, Kalecki (1954) and Robinson (1965) introduced, for the first time, the idea of integrating dynamic elements into a single model. However, Goodwin (1967) was the first to apply a genuine dynamic analysis featuring capital accumulation and employment in a predator–prey scheme. More specifically, he managed to integrate important aspects of the Harrod–Domar growth model with the Phillips curve and in so doing to generate endogenous cycles in the level of economic activity, as these are reflected in the movement of the strategic economic variables; namely, output, employment, and wages. In most of the follow up models, we cannot discern any that mimics, satisfactorily well, the actual process of capital accumulation. On the contrary, these models, by and large, rely on short–run dynamics and market clearance seeking to describe long–run tendencies and "disequilibrium" strangely enough in a–historical framework (Boldrin *et al.* 2001; Galor 2011; Fatás–Villafranca, Jarne, and Sánchez–Chóliz 2012).

From the broad classical perspective, Greiner and Semmler (1996) and Blatt (2019), by introducing Keynesian assumptions, offer short–run solutions while Desai (1973), Korpinen (1987), and Jarsulic (1991) discuss the effects of monetary policy, credit, and inflation on the growth cycle and conclude that equilibrium is unattainable. The general feature of all these models is that they do not integrate the internally generated growth and cycles into a single model. However, the Great Recession (2008–2009) and the slowdown in economic activity up until this writing show overwhelmingly that economic downturns tend to persist for long after a deep recession.

Consequently, they rendered compelling the need for the joint theorization of both economic growth and long cycles.

The salient feature of our study is the theorization of the relationship between long cycles cum economic growth by exploring their interaction, rather than by examining them in line with the conventional traditional dichotomy of neoclassical economics. In recent decades, within the broad classical tradition, there are many studies in the direction of a unified model of growth and cycles. For example, Glombowski and Krüger (1987), in dealing with various forms of technological change, examine their effects on employment, wage share, labor productivity, and capital/output ratio. Sato (1985) combines Goodwin's conceptualization of the labor market dynamics with Marx's schemes of reproduction and discusses the stability and viability of the equilibrium properties of his model. Shaikh (1989) focuses on effective demand and attempts a synthesis of classical and Keynesian ideas while Shaikh (2016) grapples with other related issues and models the unemployment rate in an alternative explanation of the Phillips curve, which he also subjects to empirical testing. Semmler (1984), Goodwin (1990), and Flaschel (2008) introduce Schumpeter's ideas on technological change to provide a synthesis in line with the Classical/Marxian and Schumpeterian traditions. Sasaki (2013), following post–Keynesian and Kaleckian ideas, introduces the effects of capacity utilization and profit share linking capitalists' propensity to invest on the level of employment. It is important to note that most Goodwin type models usually downplay the importance of fundamental aspects of the Classical and Marxian tradition, such as the law of the falling rate of profit and some other related to that features of capital accumulation, and instead place their emphasis on variables like income distribution, technology, and investment. However, the treatment of such key variables is derived not through their interactions within the same model but rather as isolated aspects of alternative models. Finally, Chatzarakis and Tsaliki (2021) utilize a dynamic system to explore the interaction of the organic composition of capital (OCC) and the rate of profit (RoP), and they show that the rise of the former leads to the decline of the latter, which sets the system to an overaccumulation mode and profitability crisis.

In this article, we develop a model designed to capture the dynamics of economic growth and cycles by incorporating into the analysis the interaction of profitability, investment, employment, technological change, and capital devaluation. The remainder of the article is structured as follows: Section 2 outlines the premises of Marx's theory of economic growth and formulates them in a system of differential equations. Section 3 presents and analyzes, qualitatively and through simulations, the interaction of profitability with the four key variables, namely, investment, employment, technological change, and devaluation of capital. Finally, section 4 concludes and makes some remarks for future research efforts.

2. "Stylized Facts" in Marx's Theory of Economic Growth

In what follows, our focus is on what we characterize "stylized facts" in Marx's theory of economic growth; that is, the rising OCC and rate of surplus value, which in combination lead to a falling RoP. Associated with these are the technological change and rising mechanization, which give rise to devaluation of capital, the cyclical movement in the reserve army of labor, and the relationship between saving and investment.[1]

[1] For the symbols used in the paper see appendix A.

2.1 The dynamics of capital accumulation and profitability

The definition of OCC is very difficult to operationalize and its details as well as intricacies have been analyzed in Shaikh (1990). In order to have a functional definition of the capital intensity or a proxy for Marx's OCC as discussed in the Schemes of Expanded Reproduction (SER) (*Capital* 2: 509–514), we employ the following formula:

$$\gamma = \frac{C}{C+V} \tag{1}$$

where γ stands for a proxy to OCC, C is the constant capital and V is the variable capital. In this formulation, we essentially normalize the OCC so as its maximum; that is, in the hypothetical case in which "workers live on thin air" or the complete mechanization of the labor process, is equal to one. In both cases, wages are equal to zero, and the OCC reaches its maximum, is equal to one. The idea behind this selection is to help us identify the complex interrelations among the variables involved. The OCC increases over time because of competition between capitals, which eventually leads to technological change and the displacement of the old by the new capital. The introduction of more fixed capital increases the OCC and the productivity of labor, which decreases the average cost and makes possible the reduction in price and the expansion of market share for those firms that survive the competition. The rising tendency of OCC is a "stylized fact" of what Marx calls the "General Law of the Capital Accumulation" which over time is manifested by the rising labor productivity and rate of surplus value, both being the result of the increasing mechanization of the labor process.

The rising OCC is the result of the self–expanded inner nature of capital, which is driven by the profit motive. According to Marx "[s]*ince the ratio of the mass of surplus value to the value of the invested total capital forms the rate of profit, this rate must constantly fall*" (*Capital* 3: 154). This is Marx's "General Law of the Falling Tendency of the Rate of Profit." which in combination with the general law of capital accumulation constitutes the fundamental macroeconomic laws that govern a capitalist economy and shape its evolution. We want to stress from the beginning the negative effect of the OCC on the RoP. This is the reason why we start with a circulating capital model reminiscent of Marx's SER and the RoP, π, or rather the profit margin on costs. In this way, we can disentangle the interaction of the relevant variables shaping the economy's movement in the very long run. The RoP employed in most empirical studies, including those dealing with long cycles, is introduced in the next paragraphs and includes both the profit margin on costs, as well as the OCC, γ. Hence, we introduce the RoP on cost, π, defined as follows:

$$\pi = \frac{S}{C+V} = \frac{S}{V}\left(\frac{V}{C+V} + \frac{C}{C+V} - \frac{C}{C+V}\right) = e(1-\gamma) \qquad (2)$$

where S is the surplus value and $e = {}^{S}/_{V}$ is the rate of surplus value. From equation (2) it follows that π is directly related to e and inversely to γ.[2] The growth rate of constant capital, g_C, is defined as:[3]

$$g_C = \hat{C} = \frac{\dot{C}}{C} = s_C\frac{\pi}{\gamma} - \delta$$

[2] The long–run falling tendency in the RoP can be shown by taking its elasticities with respect to the rate of surplus value and OCC. That is, $\frac{\partial\pi}{\partial e}\frac{e}{\pi} = (1-\gamma)\frac{e}{e(1-\gamma)} = 1$ and $\frac{\partial\pi}{\partial\gamma}\frac{\gamma}{\pi} = -\frac{\gamma}{(1-\gamma)} = -\frac{C}{V}$, respectively. For $C > V$, it follows that the effect of OCC on profitability is higher than that of the rate of surplus value leading to a long–run falling RoP. The same results with respect to elasticities are derived from the definition of the rate of profit as $r = S/C = (S/Y)/(C/Y)$ that we utilize below, where $Y = S + V$ (Tsoulfidis 2017). For further discussion and empirical results on the evolution of the RoP for the United States and other economies see Shaikh (1992 and 2016), Tsoulfidis and Paitaridis (2019), Tsoulfidis and Tsaliki (2014 and 2019).

[3] A hat over a variable denotes its growth rate while a dot denotes its time rate of change; furthermore, we set $dt \cong \Delta t = 1$ year.

where:

$$s_C = \frac{\Delta C}{S} \tag{3}$$

stands for the share of surplus value that potentially can be invested in constant capital. The idea for the inclusion of the devaluation rate, δ, is that some money must be set aside to replace the fixed capital which gradually loses value. Similarly, the growth rate of variable capital, g_V, is the investment in variable capital plus the rate of technological change, τ, thus we have:

$$g_V = \hat{V} = \frac{\dot{V}}{V} = s_V \frac{\pi}{1-\gamma} + \tau$$

where:

$$s_V = \frac{\Delta V}{S} \tag{4}$$

is the share of surplus value that potentially can be invested in variable capital and it is no different than the change in employment; τ stands for the technological change defined as the rate of change in labor productivity.

The two differential equations determine the combined dynamics of the OCC and RoP and form the following system of equations that expresses the dynamic interaction between the OCC and RoP (for details see appendix B1):

$$\begin{aligned} \hat{\gamma} &= \frac{\dot{\gamma}}{\gamma} = s_C \frac{1-\gamma}{\gamma}\pi - s_V\pi - (1-\gamma)(\delta+\tau) \\ \hat{\pi} &= \frac{\dot{\pi}}{\pi} = \hat{e} - s_C\pi + s_V \frac{\gamma}{1-\gamma}\pi + \gamma(\delta+\tau) \end{aligned} \tag{5}$$

As long as s_C, δ and τ remain positive (to preserve their economic meaning),[4] the trajectories of system (5) are attracted to a growth path, described by the following equation:

$$s_C \frac{\pi}{\gamma} - \delta = s_V \frac{\pi}{1-\gamma} + \tau + \hat{e} \tag{6}$$

According to equation (6) as with Marx's SER, a constant rate of surplus value ($\hat{e} = 0$) results in a steady–state growth path quite similar to the Harrodian warranted growth path. However, if e and s_V change, we end up with three distinct cases:[5]

- A rising $\hat{e}$, and falling s_V drive the system to a stationary state equilibrium, where $\gamma^* = 1$ (full mechanization of the production process) and $\pi^* = 0$ (zero net investment).
- Both e and s_V are rising and bringing the OCC to a limit at which the RoP increases without bounds and the system is driven further and further away from equilibrium.
- A negative $\hat{e}$ implies an increase in the wage share greater than that of the profit share driving the economy to an equilibrium akin to the Smithian "rude stage of society." where there are only self–employed workers and no capital; thus, we have $\gamma^* = 0$ and $\pi^* = 0$.

The last two cases are in Marx's list of countertendencies and, as such, their effect evaporates over time. In contrast, the first case is fully consistent with the nature of capital, and it has been repeatedly empirically long–established for many countries.

[4] We do not exclude negative technological change as in the "dark ages", although they were not as dark as they are habitually thought, at least, concerning technology. Negative technological change is rare (*e.g.*, wars, environmental or other disasters). However, these phenomena have been transitory, and the internal logic of capitalism is to expand in the long run through technological change, that is, the means to acquire profits as a purpose in itself.

[5] These results are obtained by solving equation (5), where the changes in the sign of $\hat{e}$ and s_V shape the behavior of the solutions. Chatzarakis and Tsaliki (2021: 153—157) discuss in more detail these three distinct cases.

Marx's law of the falling tendency of the rate of profit shapes the path not only of capital accumulation but also tracks down the conditions of over accumulation and economic crisis (*Capital* 3: 177). Profitability is the variable that regulates the upturns and downturns of the capitalist production by providing the motivation for innovative investment and forming the material basis for rising workers' demands for higher wages (Shaikh 2016; Tsoulfidis and Tsaliki 2019).

The system (5) can be simplified by redefining the rate of profit, r, to become comparable to the usual empirical studies for the estimation of the profit rate maintaining, at the same time, the fundamental macroeconomic variables, and their features. Thus, we may write:

$$r = \frac{S}{K} = \frac{S/\varphi(C+V)}{\varphi C/\varphi(C+V)} = \frac{\pi}{\gamma\varphi} \tag{7}$$

where $K = \varphi\, C$ the fixed capital and φ the stocks–over–flows ratio, which is found to be more or less constant over time.[6] Moreover, π captures the short–run analysis found in SER, while r is designed to capture the long–run movement of capital accumulation. Nevertheless, the fundamental properties in the definitions of the rate of profit, π and r, are the same; a rising OCC leads to a fall of both π and r; however, this inverse relation is more general and straightforwardly expressed in the latter. The dynamics can be described by a reverse logistic equation derived by taking the growth rate of equation (7):

[6] In fact, we estimated the ratio of capital stock to the intermediate inputs and wages of the US economy from 1995 to 2009 (source: WIOD 2013) and we found that this ratio does not vary by much (from 4.216 to 4.129) with slight fluctuations in the years between, while the results for the period 2000—2014 (WIOD 2016) were quite similar. We can therefore safely argue that our φ symbol pretty much can be assumed as a very slowly changing variable and for all practical purposes may be treated parametrically. Consequently, the variables s_C and g_C capture, at the same time, both the changes in C and K.

$$\hat{r} = \hat{\pi} - \hat{\gamma}$$

In combination with the differential equations of system (5), this relation gives:[7]

$$\hat{r} = \frac{\dot{r}}{r} = -(s_C - s_V)r + (\delta + \tau + \hat{e})$$

According to *Capital* 1, the change in the rate of surplus value, $\hat{e}$, is inversely related to s_V.[8] The idea is that with the progress of capitalism more and more of surplus value is spent on fixed capital and, therefore, less remains to be spent on variable capital, and so the share of variable capital diminishes as a condition for the rising productivity and falling unit cost of production. Consequently, we may rewrite the above equation as:

$$\hat{r} = \frac{\dot{r}}{r} = -a_1(s_C - s_V)r + a_2(\delta + \tau - s_V) \quad (8)$$

where a_1 and a_2 are relatively small positive parameters that help us to scale r appropriately. Equation (8) holds when $a_2 < a_1 \ll 1$, so that the RoP will fall as long as $s_C > s_V > 0$ or $s_V < 0$.[9] It is worth noting that the term $(s_C - s_V)r$ measures the difference of the actual investment in constant (or fixed) and variable capital, while s_V, δ and τ in the second term in parenthesis account for the distributional, devaluation and technological effects on profitability, respectively. In that way, a_1 and a_2 may be seen as the elasticities of the rate of profit with respect to technological and distributional parameters, respectively.

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[7] See appendix B2 for a detailed proof.

[8] This substitution is the usual practice applied in similar studies. For instance, Glombowski (1983) sets $\hat{e} = c_0 - c_1 s_V$, where c_0 and c_1 are small positive constants. In our case, we simplify the analysis by setting $\hat{e} \approx -s_V$ and in so doing absorb any difference in the scaling by dispending with the parameters a_1 and a_2.

[9] To formulate properly Marx's argument on accumulation and the falling tendency of the RoP, some further conditions must be added on system (5) and subsequently on equation (8). First, $s_C \geq 0$ indicating that the gross capital formation is always positive; second, $s_C + s_V \leq 1$ so that the economy is selfsustained; and third, $s_C > s_V$ so that capital formation exceeds the rise of employment, which implies the introduction of capital using and labor saving techniques.

2.2 The reserve army of labor

The mechanization of the production process leads to a rising OCC and increases the productivity of labor, which, in turn, reduces the unit value of commodities. Consequently, the value of commodities that workers purchase with their money wage falls and with that the variable capital decreases. An additional reason for the reduction in variable capital is the displacement of labor by machines and, therefore, the rising unemployment rate. The latter, if measured appropriately, displays a slowly rising trend with the progress of capitalism as this can be judged by the neologism "jobless growth." which is a *prima facie* recognition that employment does not necessarily keep pace with economic growth (Katrakilidis and Tsaliki 2008; Alexiou and Tsaliki 2009). Marx argued that "*it is capitalistic accumulation itself that constantly produces, and produces in the direct ratio of its own energy and extent, a relatively redundant population of laborers, i.e., a population of greater extent than suffices for the average needs of the self–expansion of capital, and therefore a surplus population*" (*Capital* 1: 443), and at the same time "[*i*]*ndependently of the limits of the actual increase of population, it creates, for the changing needs of the self–expansion of capital, a mass of human material always ready for exploitation*" (*Capital* 1: 444). In other words, the accumulation of capital in and of itself gives rise to the formation of the reserve army of labor, which, on the one hand, exerts downward pressure on wages and, on the other hand, through the mechanization enables the further division of labor, disciplines workers and increases their productivity. Hence, lower wages, higher productivity and a more disciplined labor force increase profitability and enhance the process of capital accumulation (Botwinick 1993; Tsaliki 2009). Similarly, the dynamics of employment are paced by the evolution of OCC and RoP; as we explained above, a falling share of investment in variable capital reflects the incessant mechanization of the production process.

Goodwin (1967) was the first to endogenize the dynamics of employment in a simple Harrod–like growth framework in an attempt to model the fluctuations of the reserve army of labor and their effect on destabilizing the steady–state growth path. In subsequent models (Desai 1973; Semmler 1984; Glombowski and Krüger 1987; Goodwin 1990; Flaschel 2008), the evolution of employment was examined in relation to the wage share, technological change, and capital/output ratio. The key idea permeating all the above models is that as profitability rises beyond a certain threshold, capital accumulation accelerates and so employment rises and the economy finds itself on its upturn phase; the converse will be true, if profitability falls below this threshold. Goodwin identified this threshold in the Harrodian "natural" growth rate as the growth rate of employment:

$$g_n = n + \tau$$

where, n, is the growth rate of population and τ the rate of technological change. As long as the "actual" growth rate of capital accumulation:

$$g_C = s_C r - \delta$$

differs from g_n, the employment diverges from its equilibrium state.

The idea of fluctuating employment trailing the movement in capital accumulation is in the right direction. However, most of the literature deals with the fixed effect of technological change on productivity and, in so doing, limit the analysis to the distributional aspects and the circulation sphere of the economy. Glombowski and Krüger (1987) are in deviation from this literature because in their analysis take into account the dynamics of capital–output ratios as well as productivity; in a similar vein, Goodwin (1990) addresses the issue of the diffusion of innovations. Nevertheless, a complete picture of the movement in employment or what amounts to the "industrial reserve army of labor" should include the dynamics of technological change and the way in which it affects profitability. Moreover, it is important to emphasize that the technological change in Marxian analysis is capital using and labor saving. Following Goodwin's (1967) seminal work, we stipulate that the movement in employment follows the difference between g_C from g_n. As s_V by definition (equation (4)) is the share of surplus value invested in variable capital and it is no different from the rate of change in employment, the dynamics of employment can be represented by the following differential equation (see appendix B3 for the proof):

$$\dot{s}_V = s_C r - \delta - \tau - n \quad (9)$$

Equation (9) shows that employment increases, whenever profitability is high enough so as to ensure that:

$$g_C = s_C r - \delta > \tau + n = g_n$$

indicating that the growth rate in capital accumulation is greater than the economy's natural growth rate.

In Goodwin's model, the dynamics of the employment rate combined with those of the wage share give rise to the interaction between those two quantities in a manner similar to the prey–predator scheme; that is, employment "predates" on wages generating a cyclical movement of the two variables. However, this model relies on Harrodian growth theory, which eventually veils the fact that the driving force of cycles is the accumulation of capital. From a short–run perspective, Goodwin's model is proved quite effective in describing the cyclical nature of capitalist dynamics. However, if the time span is long enough the immanent "laws of motion" of capitalist dynamics shaped by the interplay of profitability and growth should be accounted for. The reason is that as Marx explains "*... the rise of wages therefore is confined within limits that not only leave intact the foundations of the capitalistic system, but also secure its reproduction on a progressive scale*" (*Capital* 1: 537). This is equivalent to saying that the wage share is a dependent variable while profitability and growth are the independent ones; that is, the variables that set in motion the course of the entire system. From the above it follows that the evolution of the wage share can always be estimated from the movement of the independent variables of the system, that is, profitability and growth.

2.3 Saving, investment and capital accumulation

Marx does not have a fully developed theory of effective demand; he does not even raise the question except for some hints, here and there, especially in the schemes of reproduction (*Capital* 2). However, his analysis contains the required elements to develop such a theory of effective demand; in particular, by setting the conditions for balanced growth in his SER (*Capital* 2), where aggregate demand and supply are put into action by the following equilibrium conditions:

$$\underbrace{C_{\mathrm{I}} + V_{\mathrm{I}} + S_{\mathrm{I}}}_{\text{aggregate supply}} = \underbrace{C_{\mathrm{I}} + \Delta C_{\mathrm{I}} + C_{\mathrm{II}} + \Delta C_{\mathrm{II}}}_{\text{aggregate demand}}$$

where subscripts denote the two Departments and ΔC the formation of new capital (for details see Trigg 2006: chs. 3 and 5; Tsoulfidis and Tsaliki 2019: ch. 2). This equilibrium condition is rarely met at any particular time, since surplus value is not always and automatically capitalized. Following the pace of capital accumulation, surplus value is either invested, when the prospects for profits are promising, and in so doing, enhances the production of new surplus value, or it is hoarded leaking out of the circuit of capital; thereby, diminishing the potential for the creation of new surplus value. Thus, in modeling economic growth one should take into consideration that profits and, by extent, savings are not automatically invested, as in Say's law, or through variation in interest rate, as in neoclassical economics. Savings through hoarding may lead to a mismatch between aggregate demand and supply potentially triggering an economic crisis.

We hypothesize that investment depends on the availability of new techniques, τ, so that the units of capital adopting them manage to enhance productivity of labor, to reduce unit costs making possible through the undercutting of price, to eliminate competitors and expand their market share rendering credible the anticipations for higher profitability. On the other hand, hoarding is related to the cost of investing in variable capital, $s_V r$, which is the share of profits invested in wages. The following differential equation describes this relation:

$$\widehat{s_C} = \frac{\dot{s}_C}{s_C} = (\tau - s_V r) \qquad (10)$$

according to which so long as $s_V r > \tau$, that is, the cost of investing in variable capital exceeds the improvements in productivity and reductions in unit costs associated with the technological change, τ, there is no motivation whatsoever to invest in new techniques and thus hoarding becomes the next best available option. In contrast, when $\tau > s_V r$, labor saving techniques are available and investment in fixed capital increases.

2.4 Devaluation of capital

According to Marx, the speed and intensity of capital devaluation depend on the productivity of labor, which is induced by technological change (*Capital* 3: 83). When capital accumulation increases to an extent that threatens the reproduction of the entire system, the stock of capital is disposed of and, if possible, replaced; that is, capital devaluates rapidly and massively. The fixation of the rate of depreciation does not fully account for the extent of the devaluation of capital dependent upon technological change and its diffusion throughout the economy. As the OCC rises at a rate superseding that of the rising rate of surplus value and the RoP falls, the economy enters a downturn phase, and capitalists, upon extinction, are forced to "choose" new and more efficient techniques of production. Consequently, the devaluation and replacement of the old capital takes place sooner than expected according to its nominal or accounting rate of depreciation. The idea is that so long as the economy is growing, there is no pressing need on businesses for the introduction of new innovations. In the downturn of economic activity, where businesses are facing more challenging issues and their sheer survival might be at stake, the pressure to innovate increases to the point that it becomes compelling. This feature is stressed particularly by Kondratiev (1926: 39–42) but also by Schumpeter's (1942: ch. 8) "swarm of innovations" which are introduced, by and large, in the depressionary stage of the economy and by so doing become the fuel in the engine of economic growth. On the other hand, Gordon's (1980) "social structure accumulation" integrate the reorganization of the labor process to the newly created requirements for the accumulation of capital and lasting economic growth. According to Schumpeter (1939: 169–172), the length of each cycle depends on the type and relative importance of the innovations; hence the innovations that change drastically the infrastructure, equipment and organization of the labor process are related to the Kondratiev long cycles. The empirical evidence with respect to basic innovations lends support to the view that they come in swarms during the depressionary state of the long cycle of the economy (Kleinknecht 2016; Tsoulfidis and Papageorgiou 2019). It is important to stress that Marx is among the very few economists who paid particular attention to the devaluation of capital as a basic feature of the reorganization of the production process and reduction of unit cost induced by technological change.

From the above it follows that a growth model should be designed in a way such that to account for the trajectory of capital devaluation during the long cycle with two distinct components. The first refers to the ordinary process of devaluation over time and it can be depicted by a linear trend b_0. The second or specific component refers to the devaluation of capital that depends on the phase of capital accumulation. If the threshold growth rate, g_w, falls short of the $s_c r$, ($g_w < s_c r$) the devaluation rate decreases while the economy keeps growing in its upturn phase during which the motivation to innovate is relatively weak. The converse is true when, $g_w > s_c r$; the economy enters in its recessionary stage making imperative the introduction of innovations which accelerate the process of capital devaluation. Innovations are introduced massively in this stage because the risk involved in them is lower than the risk of default. The threshold g_w is akin to Harrod's "warranted" growth rate according to which aggregate demand equals aggregate supply and the utilization of capacity is therefore at its normal level, a hypothesis consistent with Marx's analysis of *Capital* 3 (pp. 189–190). The following differential equation is designed to capture these developments:

$$\dot{\delta} = b_0 + \delta(g_w - s_c r) \quad (11)$$

In other words, equation (11) describes the way in which technological change is introduced through the rate of capital accumulation and the concomitant devaluation. If $s_c r$ is relatively low, the devaluation of capital increases setting the stage for a new phase of capital accumulation to begin. The latter, by and large, is associated with the introduction of labor–saving techniques, which lower the per unit cost of production, increase productivity and eventually increase the RoP paving the way for a new rising phase of capital accumulation.

2.5 Technological change and mechanization of production process

Marx's view on innovation and technological change is based on profit seeking capitals, whose competition with each other drives them to introduce new capital–intensive techniques that lead to rising mechanization of production and labor process, increase in productivity and decreasing costs and eventually in lower prices. Mechanization is the way through which the units of capital survive in real competition and, in general, it is a built–in mechanism characterizing the capitalist mode of production.[10] Furthermore, profitability is the motive behind mechanization, not the innovative spirit of heroic entrepreneurs, and shapes both the cyclical but upward trending path of capital accumulation and the reserve army of labor (Tsaliki 2006). The two cycles are intertwined because of the introduction of labor–saving techniques. Higher investment in the prospect of higher profitability accelerates the devaluation of current capital and forces the adoption of new labor–saving techniques that lead to a rising reserve army of labor; eventually, profitability will increase, and the necessary conditions for a new phase of accumulation will be in place.

Having made the profit rate our key economic variable, we can explain the dynamics of capital accumulation as well as the movement of the other related variables. Starting by hypothesizing the economy in crisis, where the RoP is too low and many firms are in the brink of default, the devaluation rate δ exceeds a threshold and massive technological changes take place ($\dot{\tau} > 0$) until the attainment of a maximum. The accelerated devaluation of capital enhances profitability and, in so doing, accelerates investment setting up the stage for a new phase of capital accumulation.

[10] We do not exclude cases where business schemes with active workers' participation may improve efficiency. A referee pointed the example of Mondragon in Spain, a workers' cooperative with an extremely good performance record showing that people and not machines (alone) may make the difference in the competitive struggle for survival. Today at least twelve percent of world population is employed in one of the three million cooperatives https://www.ica.coop/en/cooperatives/facts–and-figures. However, we must note that this form of workers organization faces too many challenges, which make their further expansion quite difficult.

Tsoulfidis and Papageorgiou (2019), by utilizing actual data of the number of basic innovations in a long cycle framework have shown that a logistic curve describes quite accurately the process of technological change, which is more intense during the depressionary stage of the long cycle and so is the devaluation of capital. The proposed Error Function (Erf), introduced in the following differential equation, purports to model this expected and empirically ascertained regularity:

$$\dot{\tau} = b_1 \mathrm{Erf}(\delta - b_2) \tag{12}$$

where b_2 is the threshold and b_1 is a small positive constant (reaction coefficient) that help us scale appropriately the equation whose behavior is depicted in figure 1.

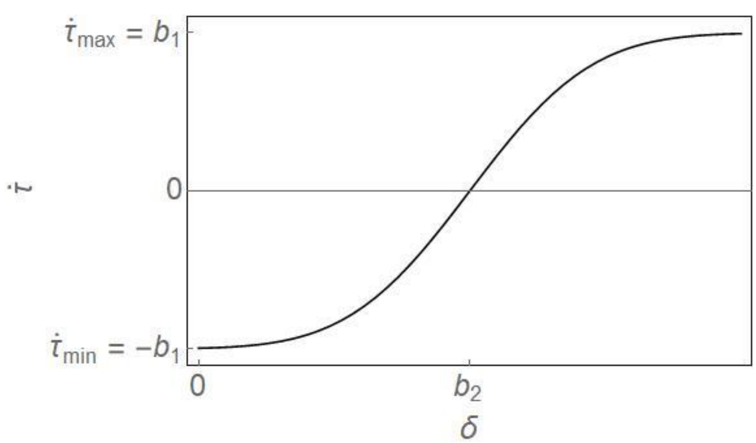


Figure 1: The Error Function of equation (12)

In figure 1, we observe that so long as $\delta < b_2$, the pace of introduction of technological change is in the negative area and following the discussion in section 2.1, we arrive at a stage of stagnant productivity. So long as $\delta > b_2$, the pace of introduction of technological change becomes positive and signifies rising productivity. It goes without saying that if $\delta = b_2$, then we have a constant τ, which takes us to the usual assumptions of the standard growth models, which hypothesize both constant rate of devaluation as well as of technological change. Moreover, from figure 1, we observe that $-b_1 < \dot{\tau} < b_1$, which is another way to say that technological change is bounded between two limits; that is, the lower and upper asymptotes of the error function, which are reflected to each other and both are related to the point of overaccumulation. Consequently, the size of b_1 defines the amplitude of technological change in the sense that the higher its value the higher the potential of an economy to enter into a new growth path through the introduction of technological change.

3. Modelling the Growth Path

For the sake of simplicity, we present our growth model in three steps. Specifically, the first step deals with the interactions of investment and employment with profitability, while the second step continues with the interplay of technological progress and devaluation of capital with profitability; the third and final step integrates the effects of all the aforementioned variables into a single growth and cycle model. In so doing, we trace the equilibrium solution of the three models, and we study their behavior illustrated with simulations along with the trajectories of the growth rates of both variable and constant capital.

3.1 First model: investment, employment and profitability

The first model consists of equations (8), (9) and (10). Setting their derivatives equal to zero ($\dot{r} = \dot{s}_C = \dot{s}_V = 0$) and for $n = 0.015$, $\delta = 0.05$ and $\tau = 0.02$, we estimate the equilibrium point:

$$\left\{ \begin{array}{c} r^* = \dfrac{a_2\tau}{a_1(n+\delta) - a_2(\delta+\tau)} = \dfrac{0.02a_2}{0.065a_1 - 0.07a_2} \\ s_C^* = \dfrac{a_1(n+\delta) - a_2(\delta+\tau)}{a_2}\dfrac{n+\delta+\tau}{\tau} = 0.27625\dfrac{a_1}{a_2} - 0.2975 \\ s_V^* = \dfrac{a_1(n+\delta) - a_2(\delta+\tau)}{a_2} = 0.065\dfrac{a_1}{a_2} - 0.07 \end{array} \right\} \qquad (13)$$

where * indicates the equilibrium value of the respective variable. For $a_1 = 0.04$ and $a_2 = 0.01$,[11] the coordinates of the equilibrium point are:

$$\{r^* = 0.1053\,, \qquad s_C^* = 0.8075\,, \qquad s_V^* = 0.19\,\}$$

Hence, the variables under investigation take on values within a reasonable range, since $r^* < 1$ and $s_C^* + s_V^* \leq 1$ indicating that investment cannot exceed surplus value in the long run.

By linearizing equations (8), (9) and (10) in the neighborhood of equilibrium,[12] we arrive at the following system of equation written in matrix form:

$$\begin{bmatrix} \Delta\dot{r} \\ \Delta\dot{s}_C \\ \Delta\dot{s}_V \end{bmatrix} = \begin{bmatrix} -a_1(n+\delta) & j_{12} & j_{13} \\ -\dfrac{n+\tau+\delta}{\tau\, {j_{32}}^2} & 0 & -(n+\tau+\delta) \\ \dfrac{n+\tau+\delta}{\tau\, j_{32}} & j_{32} & 0 \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta s_C \\ \Delta s_V \end{bmatrix}$$

where:

$$j_{12} = -\frac{a_1 {a_2}^2 \tau^2}{\big(a_1(n+\delta) - a_2(\tau+\delta)\big)^2}$$

$$j_{13} = \frac{a_1 {a_2}^2 \tau^2 + {a_2}^2 \tau \big(a_1(n+\delta) - a_2(\tau+\delta)\big)}{\big(a_1(n+\delta) - a_2(\tau+\delta)\big)^2}$$

[11] The values of n, δ and τ are taken from the available in the literature statistical values (for details see Sasaki 2013). The parameters a_1 and a_2 are used for scaling the values of our variables and their possible change in these parameters results in a slight relocation of the equilibrium point without affecting qualitatively the behavior of the system. If, however, the condition $a_2 < a_1 \ll 1$ is not fulfilled, the behavior of the system would change dramatically and it will impose results, such as $s_C > 1$ and $s_V > 1$, which indicate a no self—sustained economy in the long run.

[12] According to the theory of dynamical systems, small linear perturbations around an equilibrium point provide information about the local stability properties of the system. The eigenvalues of the linearized system are the time exponents of a linearized solution in the vicinity of the equilibrium. A negative real part of the eigenvalues means attraction towards the equilibrium, a positive real part of the eigenvalues denotes repellence, while the presence of an imaginary part indicates oscillatory behavior; the respective eigenvectors denote the direction of attraction or repellence towards equilibrium according to the sign of the real part of the eigenvalue. Moreover, the real part of the eigenvalue indicates the pace of attraction, while the imaginary part measures the period of oscillations in the vicinity of equilibrium (see Verhulst 2000).

$$j_{32} = \frac{a_2\tau}{a_1(n+\delta) - a_2(\tau+\delta)}$$

The characteristic polynomial of the above linearized system is:

$$\lambda^3 + a_1(n+\delta)\lambda^2 - (n+\tau+\delta)\left((a_1\tau + a_2) - \frac{a_2(1-a_1)\tau}{a_1(n+\delta) - a_2(\tau+\delta)}\right)\lambda + a_2\tau(n+\tau+\delta) = 0$$

whose solutions, by the use of the Routh–Hurwitz criterion, are proved to have negative real parts provided that $a_1(n+\delta) > 0$ and $a_2\tau(n+\tau+\delta) > 0$, both of which are true for economically meaningful values of a_1, a_2, n, τ and δ, and:

$$\frac{(1-a_1)a_2\tau}{(a_1\tau + a_2)\big(a_1(n+\delta) - a_2(\tau+\delta)\big)} > 1$$

The latter can be true if and only if $a_1 > a_2$ and given that $\delta > n > \tau$ as is usually set. Thus, the equilibrium point is stable for the above defined economically meaningful values of the parameters.

For the values of the parameters given above, the system provides one real and two complex conjugate eigenvalues, whose real parts are negative, deeming the equilibrium a stable node–focus. The real eigenvalue is $\lambda_1 = -0.0022$ whose eigenvector is $\vec{v}_1 = \{-0.1258,\ 0.9595,\ 0.252\}$ and represents the monotonically attracting direction towards equilibrium. The complex conjugate eigenvalues are $\lambda_2 = -0.00019 + 0.0876i$ and $\lambda_3 = -0.00019 - 0.0876i$ whose respective eigenvectors are $\vec{v}_2 = \{-0.0032 - 0.0123i,\ 0.023 + 0.6933i,\ 0.7202\}$ and $\vec{v}_3 = \{-0.0032 + 0.0123i,\ 0.023 - 0.6933i,\ 0.7202\}$ and define an attracting surface; due to imaginary parts of λ_2 and λ_3, the attracting surface displays oscillating behavior.[13]

[13] Since one of the eigenvalues is real and negative, there is a direction of monotonic attraction towards the equilibrium point, denoting the behavior of a "stable node." The two complex eigenvalues with negative real parts denote damping

The real eigenvalue, λ_1, indicates the presence of a unique trajectory monotonically attracted towards the equilibrium point. This trajectory can be traced by solving equations (9) and (10) for $\dot{s}_C = \dot{s}_V = 0$ with respect to r. That is:

$$r = \frac{n + \delta + \tau}{s_C} = \frac{0.085}{s_C}$$

and:

$$r = \frac{\tau}{s_V} = \frac{0.02}{s_V}$$

From the above, it follows that there is an inverse relationship between the rate of profit and the share of surplus value invested in constant (fixed) and variable capital, which must be fulfilled in order to attain the equilibrium point. The trajectory of this relationship is structurally unstable, and the system oscillates until the attainment of its equilibrium point. The reason is that the linearization of the system of equations (8), (9) and (10) gives eigenvalues two of which are imaginary driving the solution to oscillate around the trajectory. The system can move along the trajectory only when the following conditions hold:

$$r = \frac{n + \delta + \tau}{s_C} = \frac{\tau}{s_V} \tag{14}$$

That is, s_C and s_V must change accordingly so that equation (14) holds. The conditions described in equation (14) coincide with Harrod's "knife–edge" equilibrium (Harrod 1939) and with Marx's balanced growth conditions derived from the SER (*Capital* 2). The instability of this balanced

oscillations on surfaces orthogonal to this attracting direction, that is, characteristic of a "stable focus" (see Verhulst 2000). Such an equilibrium is called "stable node—focus." as it combines the two results. It is worth noting that the attraction is expected to be slow, as the magnitudes of the eigenvalues are generally small for the range of economically meaningful values of the parameters; hence, the system will stabilize after many oscillations.

growth path reflects also Domar's "disinvestment problem" (Domar 1948) as well as Luxemburg's (1913) together with Grossman's (1929) earlier analyses on the long–run stability of equilibrium.[14]

In figure 2, the grey arrows denote the vector field of the system, the blue, orange, and green curves denote different solutions, the black curve denotes the unstable "steady–state" trajectory (equation (14)) and the red point stands for the equilibrium point. The trajectories of the system for randomly chosen various initial conditions of r, s_C and s_V oscillate while maintaining a certain long–run trend (the black solid line) towards equilibrium (red point) at the end of the skewed cone that stands for the falling tendency of the RoP. Moreover, the period of oscillations, although differs slightly depending on the initial conditions, is very specific; as obtained from the simulations, the average number of steps required for each full cycle is about 40 and each step we may set it equal to one year. A time period consistent with Kondratiev's long cycles and what historically has been observed for the United States and other major economies (Shaikh 1992 and 2016; Tsoulfidis and Papageorgiou 2019; Tsoulfidis and Tsaliki 2019).[15]

[14] It is interesting to note that to the extent this relation holds (and contains the equilibrium point), the RoP may decrease, but it never reaches zero, unless s_C and s_V explode to infinity, or the parameters n, δ and τ are set to zero.
[15] The amplitude (range) but not the period of oscillations is affected by slight changes in the values of parameters while the period of oscillations remains fast. It is worth noticing that for the cycles near the equilibrium point, their period can be obtained from the imaginary part of the eigenvalues, i, as follows $2\pi/i$; however, if the initial conditions are distant from the equilibrium, as in our case, the length of the cycle is measured approximately by the period of the simulations.

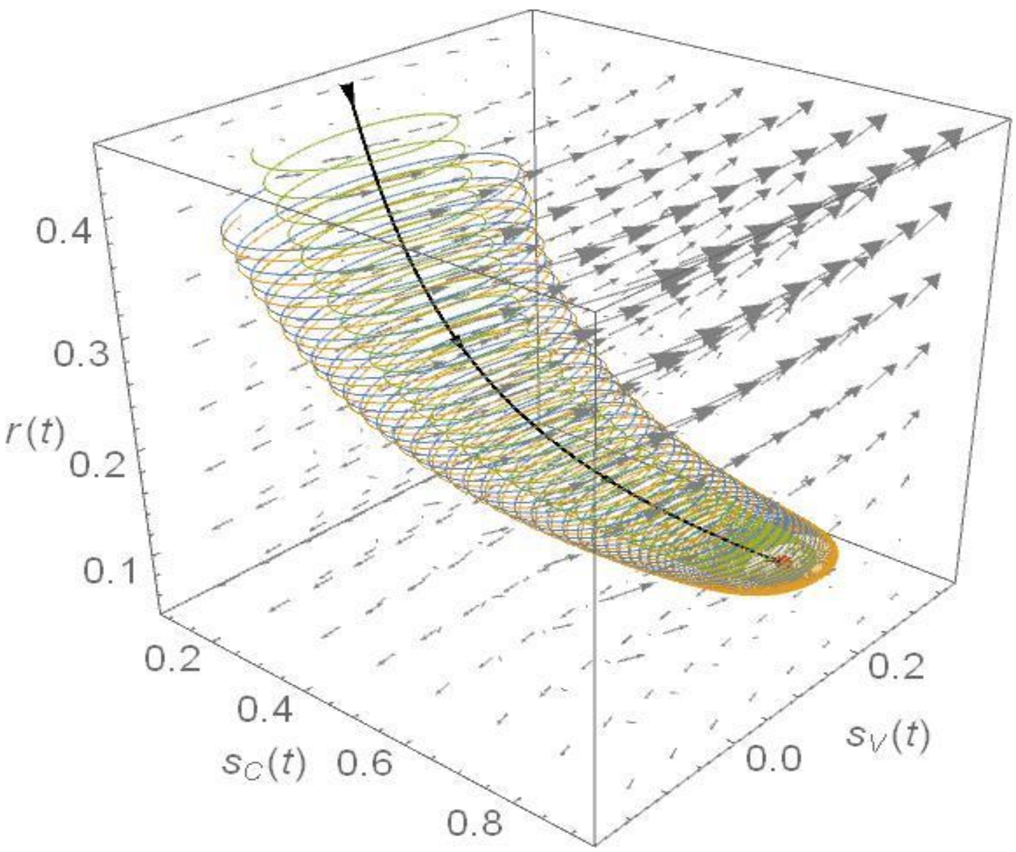


Figure 2: The phase space of equations (8), (9) and (10)

In figure 3, we present the simulated time series of s_C, s_V and r for three hundred years (steps), where the approximate 40–year cycles are clearly observed.

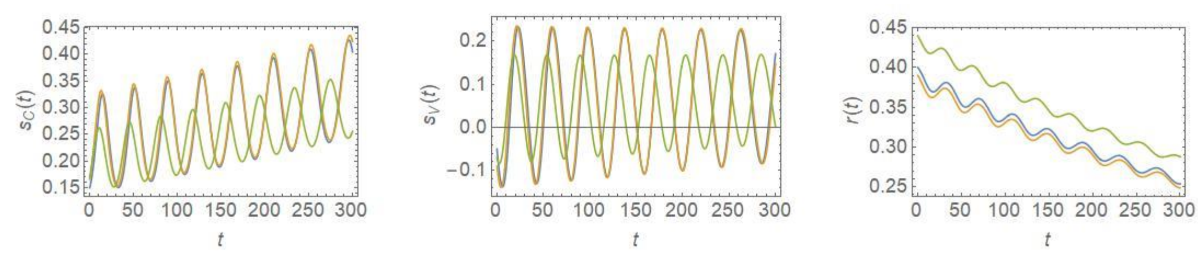


Figure 3: Time series for s_C, s_V and r derived from numerical integrations (simulations) of equations (8), (9) and (10) utilizing different initial conditions

Furthermore, in figure 3, we present three different cases with different initial conditions for each variable so as to show that their evolution remains the same. The trend in s_C is rising indicating that the mechanization increases the intensification of the labor process, the s_V oscillates reflecting the circular nature of the reserve army of unemployed while the evolution of r is as one would expect from the law of the long–run falling RoP.[16]

It is interesting to note that investment and employment oscillate around the falling trajectory of the RoP. For a constant RoP, equations (9) and (10) constitute a dynamical system for s_C and s_V whose equilibrium point is given by equation (14) and the solutions are counterclockwise oscillations around it, as stated by the lower–right 2×2 submatrix of the Jacobian. These oscillations indicate cycles where investment "leads." hence an increase (decrease) in investment leads to an increase (decrease) in employment. Following Goodwin (1967), several studies explore the presence of such cycles (*e.g.*, Flaschel, Franke, and Semmler 2007; von Armin, and Barrales 2015; Setterfield 2021) whose periods are found to be less than twenty years. The difference in the length of our cycles may be attributed to the choice of the "investment function" in equation (10), which is related to the availability of new techniques and the drastic change in both the mechanization and organization of the labor process, rather than to changes in the expectations of the investors (Schumpeter 1939).

[16] The simulations were carried out with the use of Mathematica programming language and the initial conditions as well as the code to reproduce the trajectories of variables are available on request.

3.2 Second model: technology, devaluation and profitability

Equations (8), (11) and (12) for $g_w = 0.03$, $b_2 = 0.05$ and $s_C = 0.5$ form a dynamical system that describes the interaction of technological change, devaluation of capital and profitability.[17] Setting the time derivatives on the left–hand side of the above equations equal to zero ($\dot{r} = \dot{\tau} = \dot{\delta} = 0$), we estimate the unique equilibrium point, which is:

$$\left\{\begin{array}{c} r^* = \dfrac{b_0 + b_2 g_w}{b_2 s_C} = 0.06 + 40b_0 \\ \tau^* = \dfrac{a_1}{a_2}\dfrac{(b_0 + b_2 g_w)(s_C - s_V)}{b_2 s_C} - b_2 - s_V = \dfrac{a_1}{a_2}(0.06 + 40b_0)(0.5 - s_V) - 0.05 - s_V \\ \delta^* = b_2 = 0.05 \end{array}\right\} \quad (15)$$

For $a_1 = 0.04$, $a_2 = 0.01$ and $b_0 = 0.004$,[18] the coordinates of the equilibrium point are:

$$\{r^* = 0.22\,, \qquad \tau^* = 0.39 - 1.88 s_V\,, \qquad \delta^* = 0.05\,\}$$

From the above, we observe that τ^* is a function of s_V, that is, it depends on the portion of surplus value invested on variable capital. Interestingly, if $s_V < s_C \frac{a_1 b_0 + b_2(a_1 g_w - a_2 b_2)}{a_1 b_0 + a_1 b_2 g_w + a_2 b_2 s_C} = 0.208$, then $\tau^* > 0$.[19] Hence, the condition of $s_V < 0.208$ forms an upper boundary beyond which technological change is no longer labor saving.

Subsequently, we linearize the system in the neighborhood of the equilibrium point (see relation 15) and we get:

---

[17] The value of g_w is chosen so as to be in line with the available econometric evidence (Bergeaud, Cette, and Lecat 2015), while b_2 takes on the usual values employed in growth models (see footnote 9 concerning δ). Finally, $s_C = 0.5$ is in line with the results (on an average) reported in the previous section. Suffice it to say that the overall behavior of the model is robust to variations in the parameters.

[18] For the numerical values of parameters a_1 and a_2, see footnote 9. The parameter b_0 denotes the intertemporal trend of devaluation of capital, which is slow. Our experimentations with slightly different values of b_0 resulted in a similar overall behavior of the system.

[19] For different values of s_V ($= 0.1,\ 0$ and -0.1) the τ^* takes on values ($= 0.266,\ 0.47$ and 0.674, respectively) lower than one, indicating that the growth rate of labor productivity remains within reasonable rates.

$$
\begin{bmatrix} \dot{\Delta r} \\ \dot{\Delta \tau} \\ \dot{\Delta \delta} \end{bmatrix} = \begin{bmatrix} -\frac{a_1(b_0+b_2g_w)(s_C-s_V)}{b_2s_C} & \frac{a_2(b_0+b_2g_w)}{b_2s_C} & \frac{a_2(b_0+b_2g_w)}{b_2s_C} \\ 0 & 0 & \frac{2b_1}{\sqrt{\pi}} \\ -b_2s_C & 0 & -\frac{b_0}{b_2} \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta \tau \\ \Delta \delta \end{bmatrix}
$$

from which we get one real and two complex conjugate eigenvalues. From the characteristic polynomial:

$$
\lambda^3 + \left(\frac{b_0}{b_2} + \frac{a_1(b_0+b_2g_w)}{b_2s_C}\right)\lambda^2 + \left(a_2 - \frac{a_1b_0}{b_2}\frac{s_C-s_V}{s_C}\right)(b_0-b_2g_w)\lambda + \frac{2}{\sqrt{\pi}}a_2b_1(b_0+b_2g_w) = 0
$$

and applying the Routh–Hurwitz stability criterion, we notice that all three eigenvalues have negative real parts, hence the equilibrium point is stable, when $s_C > s_V$, which is assumed to be always true, and:

$$
b_1 < \frac{\sqrt{\pi}}{2}\left(1 - \frac{a_1b_0}{a_2b_2}\frac{s_C-s_V}{s_C}\right)
$$

which is not necessarily true for all possible value of a_1, a_2, b_0 and b_2. Specifically, the real eigenvalue is always negative, denoting the existence of a monotonically attracting direction, while the sign of the real parts of the complex eigenvalues alternates. For the values of the parameters given above, the real part of the complex eigenvalues is negative when $b_1 < 0.009$, hence the system converges to equilibrium via dampening oscillations (as in figure 4), and it is positive when $b_1 > 0.009$, so the solutions diverge from the equilibrium via increasing oscillations (as in figure 6). It is worth noting that changes in s_V do not affect the eigenvalues and, therefore, the behavior of the system. In table 1, we illustrate the above by experimenting with different values for s_V and b_1.

Table 1: Properties of the equilibrium point

Parameters values	Eigenvalues	Characterization[20]
For $\boldsymbol{b_1} < 0.009$, stable equilibrium		
$b_1 = 0.004$ and $s_V = 0.1$	$\lambda_1 = -0.8126$ $\lambda_2 = -0.0113 + 0.1231i$ $\lambda_3 = -0.0113 - 0.1231i$	Stable node–focus
$b_1 = 0.004$ and $s_V = 0$	$\lambda_1 = -0.8127$ $\lambda_2 = -0.0156 + 0.1226i$ $\lambda_3 = -0.0156 - 0.1226i$	Stable node–focus
$b_1 = 0.004$ and $s_V = -0.1$	$\lambda_1 = -0.8129$ $\lambda_2 = -0.0199 + 0.1219i$ $\lambda_3 = -0.0199 - 0.1219i$	Stable node–focus
For $\boldsymbol{b_1 = 0.009}$, degenerate equilibrium		
$b_1 = 0.009$ and $s_V = 0.1$	$\lambda_1 = -0.8349$ $\lambda_2 = 0.1829i$ $\lambda_3 = -0.1829i$	Degenerate node–focus
$b_1 = 0.009$ and $s_V = 0$	$\lambda_1 = -0.8353$ $\lambda_2 = 0.1828i$ $\lambda_3 = -0.1828i$	Degenerate node–focus
$b_1 = 0.009$ and $s_V = -0.1$	$\lambda_1 = -0.8357$ $\lambda_2 = 0.1826i$ $\lambda_3 = -0.1826i$	Degenerate node–focus
For $\boldsymbol{b_1} > 0.009$, unstable equilibrium		
$b_1 = 0.01$ and $s_V = 0.1$	$\lambda_1 = -0.8392$ $\lambda_2 = 0.00198 + 0.1923i$ $\lambda_3 = 0.00198 - 0.1923i$	Generalized saddle
$b_1 = 0.01$ and $s_V = 0$	$\lambda_1 = -0.8396$ $\lambda_2 = 0.00223 + 0.1922i$ $\lambda_3 = 0.00223 - 0.1922i$	Generalized saddle
$b_1 = 0.01$ and $s_V = -0.1$	$\lambda_1 = -0.8399$ $\lambda_2 = 0.00643 + 0.1921i$ $\lambda_3 = 0.00643 - 0.1921i$	Generalized saddle

[20] For the definition of "stable node—foci" see footnote 10. "Saddles" are equilibrium points combining both stable and unstable characteristics. In this sense, any equilibrium with both attracting and repelling directions is characterized as a "saddle." If some of these directions are associated with oscillations (*e.g.*, a monotonic attraction or repellence along one direction combined with increasing or dampening oscillations orthogonal to it), then the equilibrium is known as a "generalized saddle."

The phase space of equations (8), (11) and (12) for $b_1 = 0.004$ is depicted in figure 4

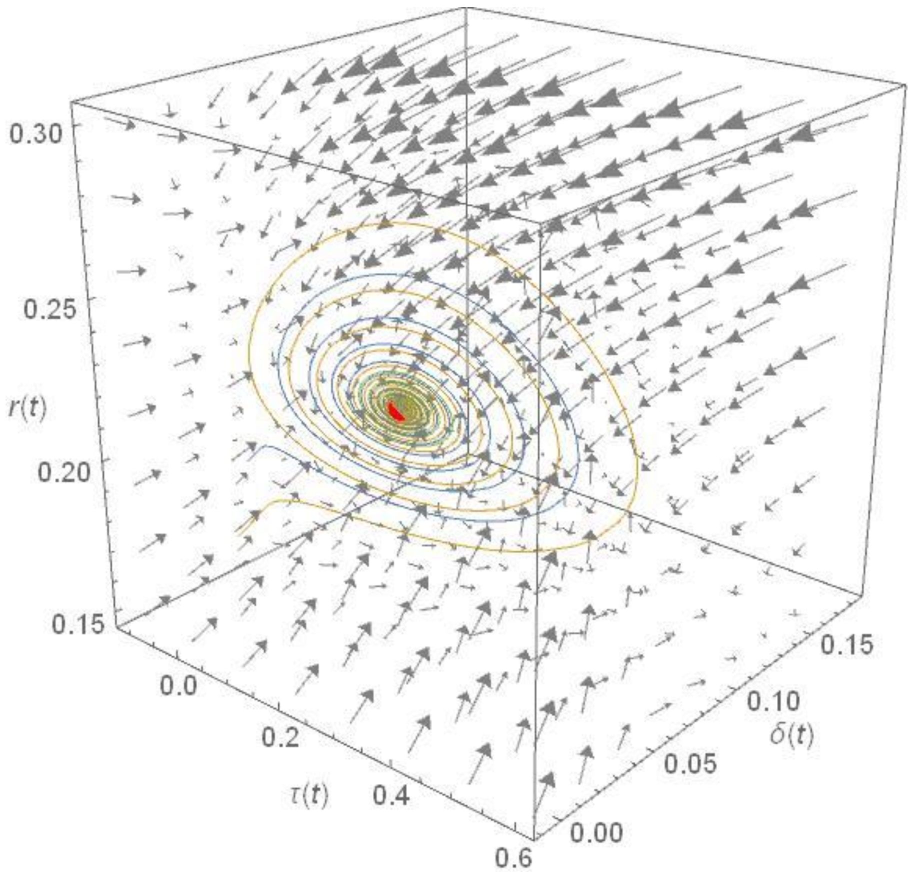


Figure 4: The phase space of equations (8), (11) and (12) for $b_1 = 0.004$.

In figure 4, the grey arrows denote the vector field of the system, that is, the direction of their trajectories; the blue, orange, and green curves denote the trajectories of the system for different initial conditions –chosen randomly; the red point stands for the equilibrium point of the system. We observe that all solutions converge to equilibrium following oscillating paths.

In figure 5, we present the simulated time series of τ, δ and r in four hundred steps or years. Given the eigenvalues in table 1, we estimate that the dampening oscillations display a period of $\frac{2\pi}{0.123} \approx 51$ years.

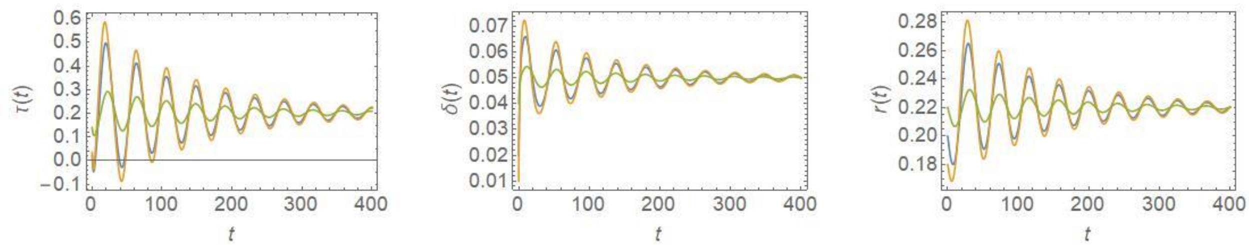


Figure 5: Time series for τ, δ and r derived from simulations of equations (8), (11) and (12) for different initial conditions and $b_1 = 0.004$.

The dampening oscillations are explained by the low value of b_1. In subsection 2.5, we argued that the phase of capital accumulation is confined between b_1 and $-b_1$ reflecting the maximum positive and negative, respectively, pace of technological change approximating its asymptotes; hence, a low $b_1 = 0.004$ indicates a low rate of introduction of new techniques in the production process and thus a slower rate of capital accumulation which fails to restart the economy. Thus, the amplitude of the cycle reduces and eventually dies out.

A more interesting case appears when the value of b_1 increases so long as the equilibrium becomes unstable (see table 1). In this case, the asymptotic stability analysis followed so far is insufficient to undress the dynamics of the system, as the nonlinearities –especially those born by the Error function in equation (12)– become significant. More specifically, when the equilibrium shifts from stable to unstable while the solutions oscillate around it, an Andronov–Hopf bifurcation occurs and it is manifested in a closed periodic solution around the equilibrium point, known as a "limit cycle" (Verhulst 2000). The existence of a "limit cycle" can be proved by means of the generalized Bendixson's criterion (Li and Muldowney 1993), according to which the infinima of:

$$X = \frac{\partial \dot{r}}{\partial r} + \frac{\partial \dot{\tau}}{\partial \tau} + \left|\frac{\partial \dot{r}}{\partial \tau}\right| + \left|\frac{\partial \dot{r}}{\partial \delta}\right| + \left|\frac{\partial \dot{\tau}}{\partial r}\right| + \left|\frac{\partial \dot{\tau}}{\partial \delta}\right|$$

$$Y = \frac{\partial \dot{r}}{\partial r} + \frac{\partial \dot{\delta}}{\partial \delta} + \left|\frac{\partial \dot{r}}{\partial \tau}\right| + \left|\frac{\partial \dot{r}}{\partial \delta}\right| + \left|\frac{\partial \dot{\delta}}{\partial r}\right| + \left|\frac{\partial \dot{\delta}}{\partial \tau}\right|$$

$$Z = \frac{\partial \dot{\tau}}{\partial \tau} + \frac{\partial \dot{\delta}}{\partial \delta} + \left|\frac{\partial \dot{\tau}}{\partial r}\right| + \left|\frac{\partial \dot{\tau}}{\partial \delta}\right| + \left|\frac{\partial \dot{\delta}}{\partial r}\right| + \left|\frac{\partial \dot{\delta}}{\partial \tau}\right|$$

must be positive (essentially, the minimum of X, Y and Z with respect to the parameters must be positive).[21] It is proved that when:

$$b_1 > \frac{\sqrt{\pi}}{2 - s_C\sqrt{\pi}} \frac{b_0}{b_2} \qquad (16)$$

the above stated criterion for positive infinima of X, Y and Z holds; this inequality depends solely on the values of b_1 so long as $s_C < {}^{2}/_{\sqrt{\pi}}$ and all other parameters (except for s_V) are positive. Subsequently, the solutions of the system converge to a "limit cycle." which can be traced via simulations of the system for several initial conditions as illustrated in figure 6.

[21] Usually, the existence of a "limit cycle" is proved by means of the Poincaré—Bendixson theorem, according to which we must consider the sign of the divergence of the dynamical system

$$\vec{\nabla}\vec{f} = \frac{\partial \dot{r}}{\partial r} + \frac{\partial \dot{\tau}}{\partial \tau} + \frac{\partial \dot{\delta}}{\partial \delta}$$

If this divergence is positive, the system is explosive (hence, solutions diverge); if it is negative, the system is dissipative (hence, solutions converge); whenever the divergence alternates sign, the behavior of the system changes from explosive to dissipative and *vice versa* (Verhulst 2000). Attractors, such as "limit cycles." lie in areas where these changes take place. However, this theorem is solidly stated only for two—dimensional dynamical systems. Hence, we are compelled to use a generalization given by Li and Muldowney (1993).

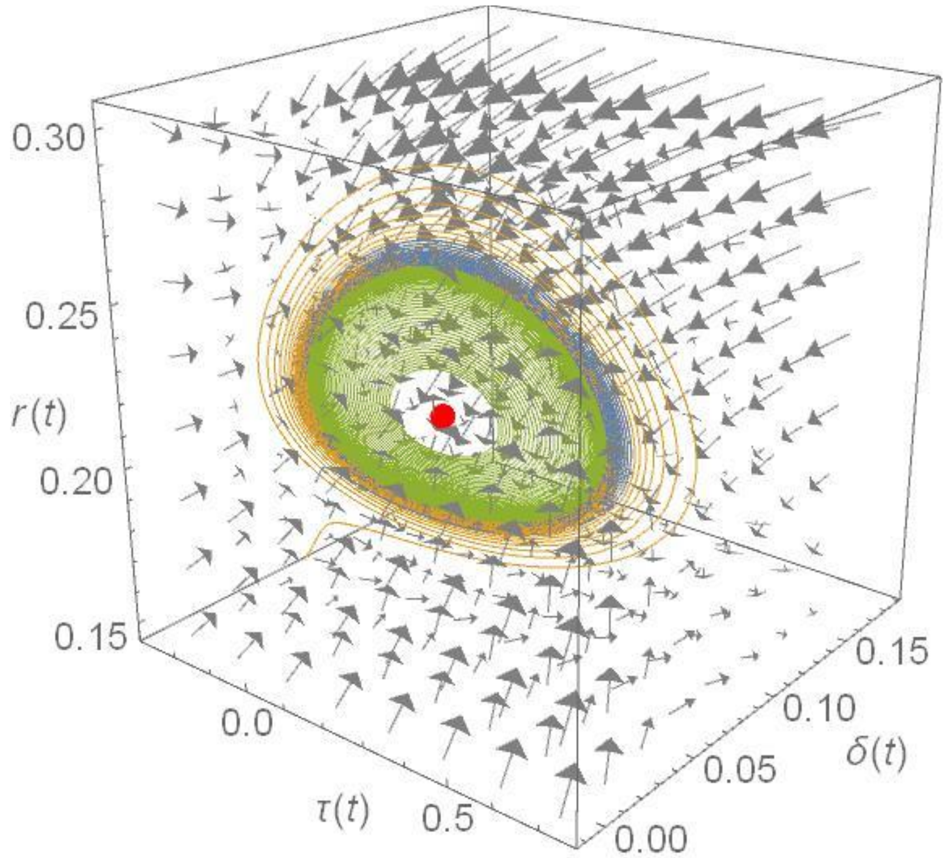


Figure 6: The phase space of Eqs. (8), (11) and (12) for $b_1 = 0.01$.

In figure 6, the grey arrows denote the vector field of the system, essentially the direction of their trajectories; the blue, orange, and green curves denote the trajectories of the system for different initial conditions while the red point, as before, stands for the equilibrium point of the system. The "limit cycle" solution is clearly visible.

In figure 7, we present the simulated solutions of τ, δ and r after four hundred steps (years). The estimated period of the oscillations close to the equilibrium is estimated by the ratio $\frac{2\pi}{0.192} \cong 33$ years but stabilizes after an almost–40–year period as it reaches the "limit cycle."

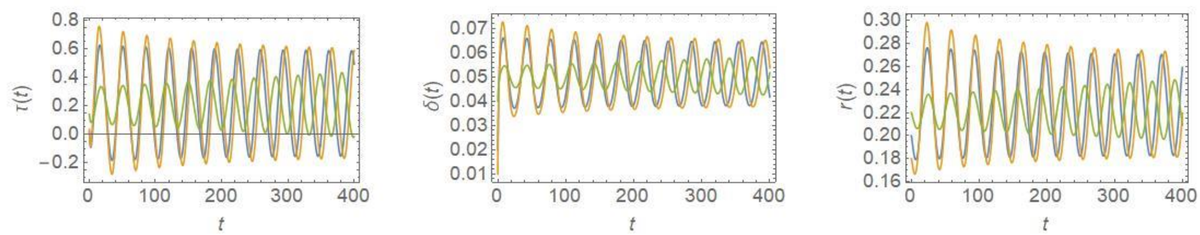


Figure 7: Time series for τ, δ and r derived from simulations of Eqs. (8), (11) and (12) for different initial conditions and $b_1 = 0.01$.

In figure 7, we observe persistent oscillations; hence, in this case the value of $b_1 = 0.01$ is large enough to introduce the necessary innovations and "reheat" the economy at the end of each cycle, so as to stimulate a new phase of accumulation.

3.3 Third model: Putting all variables together

The combined system is consisted of differential equations (8), (9), (10), (11) and (12) depicting the dynamic aspects of the variables involved in the analysis, that is, profitability, employment, investment, devaluation, and technological change, respectively. We opted to include all the above variables in our growth cum cycles model so as to capture, as comprehensively as possible, the complex dynamical nature of the capitalist economy. For the sake of simplicity, we limit the analysis to specific cases that reasonably describe the dynamic behavior of the system while using simulations we confirm its pattern. It is important to stress that we experimented with relatively small variations in the parameters of our model, and we found no qualitatively different behavior strengthening our view that our proposed model of growth and cycles is robust to parameter changes. Furthermore, economically meaningful solutions require that the values of a_1, a_2, b_0 and b_1 to be restricted to small intervals while the values for the remaining parameters are taken from those employed in the extant literature (Sasaki 2013).

Setting all derivatives of equations (8), (9), (10), (11) and (12) equal to zero, that is, $\dot{r} = \dot{s}_C = \dot{s}_V = \dot{\delta} = \dot{\tau} = 0$, we obtain the equilibrium point:

$$\left\{\begin{array}{c} r^* = \dfrac{a_2\big(b_0 - b_2(b_2 - g_w + n)\big)}{a_1 b_2(b_2 + n) - a_2\big(b_0 + b_2(g_w - n)\big)} \\ s_C^* = \dfrac{(b_0 + b_2 g_w)\Big(a_1 b_2(b_2 + n) - a_2\big(b_0 + b_2(g_w - n)\big)\Big)}{a_2 b_2(b_0 - b_2(b_2 - g_w + n))} \\ s_V^* = \dfrac{a_1}{a_2}(b_2 + n) - \dfrac{b_0}{b_2} - g_w + n \\ \tau^* = \dfrac{b_0}{b_2} - b_2 + g_w - n \\ \delta^* = b_2 \end{array}\right\} \tag{17}$$

Replacing the values of $a_1 = 0.04$, $a_2 = 0.01$, $b_0 = 0.004$, $b_1 = 0.005$,[22] $b_2 = 0.05$, $n = 0.015$ and $g_w = 0.03$ in equation (17), the coordinates of the equilibrium point are:

$$\{r^* = 0.0947\,, \quad s_C^* = 0.8708\,, \quad s_V^* = 0.1188\,, \quad \tau^* = 0.045\,, \quad \delta^* = 0.05\,\}$$

Surprisingly enough, these equilibrium points are quite close to those one expects and, in fact, finds for the US economy when it comes to the rate of profit (Shaikh 2016: 410; Tsoulfidis and Tsaliki 2019: 424; the Penn's database rate of profit denoted by IRR), to technological change (the figure is consistent with the estimates used in theoretical analysis and it is found in the empirical

[22] Since we are dealing with all variables of the system, the values of the parameters seem to be further restricted to economically meaningful equilibrium; however, the qualitative behavior of the system is consolidated and parameters, such as b_1, may lie in a wider interval than that utilized in subsection 3.2.

literature, *e.g.,* Sasaki 2013) and to depreciation (the Penn world tables give a delta for the United States and other countries not too different from our figure and also Shaikh 2016: 847). Finally, the shares of surplus value devoted to investment in constant and variable capital are not out of touch but in spirit with those found in *Capital* 2, namely 0.8 and 0.2 for constant and variable capital, respectively. We feel that our figures of 0.9 and 0.1 are more in line with the nature of capital in the present phase of capitalism. It is important to stress, at this point, that our simulations are quite robust to reasonable changes in the parameters. Equally important is to say that the imperative to "accumulate for accumulation's shake" gives rise to capital using and labor saving technological change. In the analysis of Marx and the classical economists, there is no substitutability as in the neoclassical theory, where a slight increase in wages, for example, leads to the substitution of capital for labor. Wages can increase so long as they do not interfere with the process of capital accumulation.

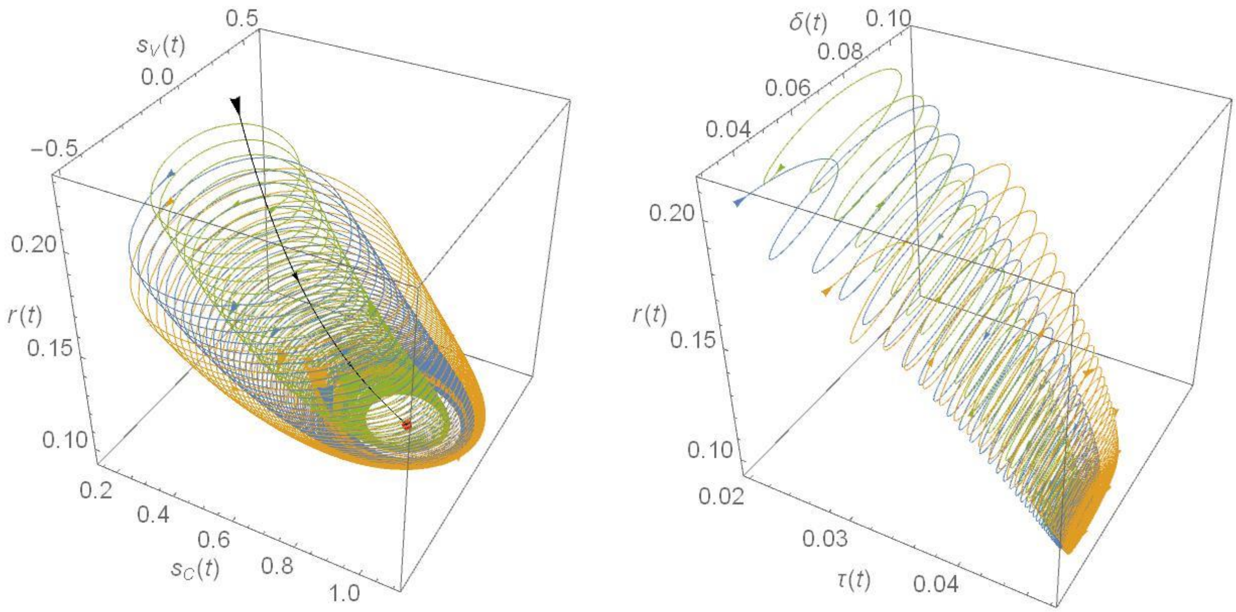
Small linear perturbations in the neighborhood of this equilibrium point yield the following linearized system:[23]

$$\begin{bmatrix} \Delta\dot{r} \\ \Delta\dot{s}_C \\ \Delta\dot{s}_V \\ \Delta\dot{\tau} \\ \Delta\dot{\delta} \end{bmatrix} = \begin{bmatrix} 0. & -0.0825 & -0.1034 & 0.2177 & 0 \\ 0.1263 & 0 & 1.1611 & -1 & 0 \\ -0.00027 & 0.00122 & -0.0021 & 0.00095 & 0.00095 \\ 0 & 0 & 0 & 0 & 0.00226 \\ -0.0063 & 0 & -0.05806 & 0 & -0.08 \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta s_C \\ \Delta s_V \\ \Delta \tau \\ \Delta \delta \end{bmatrix}$$

The corresponding eigenvalues of the system are $\lambda_1 = -0.07904$, $\lambda_2 = -0.0002 + 0.0947i$, $\lambda_3 = -0.0002 - 0.0947i$, $\lambda_4 = -0.0014 + 0.0001i$ and $\lambda_5 = -0.0014 - 0.0001i$. The five

[23] A referee of the journal suggested the Routh—Hurwitz criterion, which was applied as proposed in the previous models; however, applying this criterion in the current more complex model provides complicated and unclear results, unless we apply specific parameter values. In the interest of brevity and clarity of the presentation we opted to leave out of our analysis this particular criterion. Performing simulations for an economically meaningful range of the parameter values, we observe the same qualitative behavior of the model. One of these cases is presented in this subsection.

eigenvalues have negative real parts while four of them display an imaginary part; hence, the equilibrium is a stable node–focus. As figure 8 shows in two distinct subspaces, the solutions are attracted towards equilibrium with oscillations. The blue, orange and green curves denote trajectories of the system for different initial conditions; the red dot stands for the equilibrium point of the system.[24]



(a) (b)

Figure 8: The phase space of the complete system of equations (8), (9), (19), (11) and (12).

The above system of equations, for realistic parameter values, gives rise to a long–run tendential fall in the RoP, which is consistent with the cyclical behavior of the key variables of the system; namely, investment (capital accumulation) in both constant and variable capital, technological

[24] The five–dimensional phase space of the system cannot be shown. As a result, we present two distinct three–dimensional projections while the entire solution is the combination of them.

change, and devaluation of capital. It is worth noting that in the so generated synthetic system of differential equations, δ and τ no longer follow limit cycle patterns and they adjust to the falling tendency of the RoP in such a way that τ, as theoretically expected, increases over the long run. The simulated time series of our variables are presented in figure 9 whose salient feature is their long cycles of approximate length of 50 years.

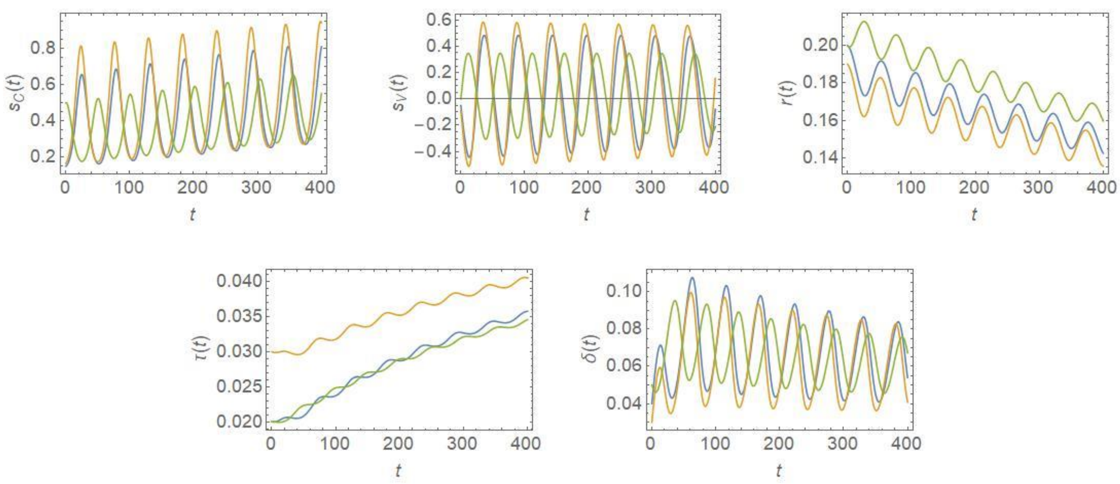


Figure 9: Time series for s_C, s_V,r, δ and τ derived from simulations of equations (8), (9), (10), (11) and (12) utilizing different initial conditions.

The trajectories of the variables displayed in each of the panel of five graphs in figure 9 are consistent with the anticipated of the classical/Marxian theory of the evolution of capitalism. More specifically, our model conveys the long–run tendency of the RoP to fall; this is accompanied by an increase in the ceiling of the fluctuation of s_C, which means that the share of surplus value intended for investment in fixed capital increases slightly over time as the RoP falls. We also confirm the oscillatory behavior of s_V, which during recessions takes on negative values; a result which is consistent with Goodwin's (1967) description of a fluctuating labor market, and it has been ascertained by the subsequent literature (Desai 1973; Glombowski and Krüger 1987; Flaschel 2008; Sasaki 2013) and is consistent with Marx's argument of a cyclical reserve army of labor. In regard to the long–run movement of the s_V variable, we observe that, in the long run, the employment creating and destroying effects cancel each other out. Equation (14) of the rate of profit implies an increase in s_V for constant technological change, while the increase of capital using and labor saving technological change should be higher than the decrease in r in order to ensure a decrease in s_V. Finally, we notice that the δ ends up fluctuating around the threshold. All the results are consistent with the theoretical expectations and are in line with those found in the respective empirical literature. The empirical research of Korotayev and Tsirel (2010) has shown that the long wave frequency in the world economy is superimposed over all shorter economic fluctuations. Hence, the derivation of long cycles is not only an empirical curiosum as one could argue on the basis of Kondratiev's (1926), Schumpeter's (1942) or Gordon's (1980) findings, among others but is based on solid theoretical underpinnings.

By invoking equation (14), we can shed more light on the Harrodian "knife–edge" problem. The latter, in the neoclassical and the usual Keynesian discussions is dealt with the attainment of stability through either the market mechanism or government intervention, respectively. Consequently, no adequate attention is paid to explain the nature of instability as such. In some other Keynesian approaches, the instability is mainly addressed by means of the exogenously determined entrepreneurial "animal spirits." By contrast, our model presents an analysis derived from the inner nature of capitalism and the associated with it built–in mechanisms. In figure 10, we display the movement of the growth rates for constant (fixed) and variable capital according to the following differential equations:

$$g_C = s_C r - \delta \qquad \text{and} \qquad g_V = s_V r + \tau$$

It is interesting to note that these growth rates are characterized by the 50–year long cycles while their difference, $g_C - g_V = \hat{e}$, estimated form equation (6) is also displayed in the right end of figure 10.

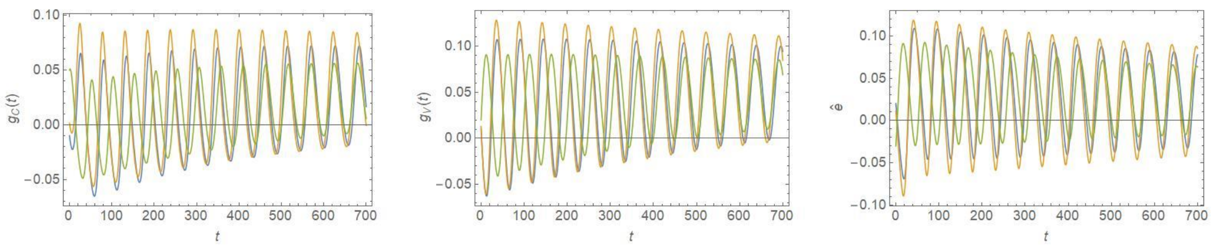


Figure 10: The estimates of g_C, g_V and $\hat{e}$, derived from simulations

The difference between g_C and g_V shows how far the system is from the "warranted" path, g_w. A positive difference indicates that $\hat{e}$ increases at a rate slower than that of g_C; hence, the OCC rises faster than e and eventually we end up with falling RoP. A negative difference reflects the counter tendencies that are at work (that have been already mentioned) and set a mechanism to restore profitability restarting a new phase of capital accumulation. Moreover, this difference reveals the fundamental mechanism underneath the so called "animal spirit" claims of Keynesian economists. It is clear from equation (6) that the term $g_C - g_V$ "corrects" for the Harrodian notion of a "steady–state" growth path by accounting for the causes and the countertendencies at work that give rise to a falling tendency of the RoP. Hence, it explains why and how constant (fixed) and variable capitals are disproportionately growing rendering the attainment of the "warranted" growth rate distant if not impossible.

It will be interesting to explore the classical economists' assumption of a "stagnant economy." according to which the rate of profit vanishes. Assuming $r = 0$ as an initial condition for the model, equation (8) guarantees that $\dot{r} = 0$; furthermore, equations (9), (10), (11) and (12) indicate $\dot{s}_C > 0, \dot{s}_V < 0, \dot{\delta} > 0$ and $\dot{\tau} > 0$, so the propensity to invest on fixed capital increases indefinitely, the propensity to invest in variable capital decreases indefinitely while the devaluation of capital and technological change rise unboundedly. This extreme state can be identified only with a fully automated economy, where labor is no longer employed in production, thus no surplus value is produced. However, the hypothetical case of zero profitability would imply zero investment and therefore the collapse of the capitalist system. However, in our model, r may tend towards, but it does not become zero unless we fix the initial conditions and our parameters take on unrealistic values.[25]

4. Conclusions

Our growth cum cycle model presented in section 3.1 showed that the dynamics of investment and employment in and of themselves are sufficient to generate both the cyclical behavior of the system and the tendential fall in the RoP. At first sight, these cycles in the short run seem to be investment led; however, their long–run dynamics are dominated by declining profitability. The model presented in section 3.2 confirmed that the dynamics of technological change and subsequent devaluation of capital generate a cyclical behavior in the economy in the form of a "limit cycle." but they do not give rise to a long–run falling tendency in the RoP. Finally, the fully integrated model presented in section 3.3 contains the dynamic interactions of all the stylized facts of capital

[25] Simulations, performed with $r = 0$ as an initial condition, produced the above—described behavior of the system.

accumulation presented in section 2. Furthermore, our model is consistent with Marx's analysis in *Capital* 1 and 2 according to which economic growth is cyclical and in the long run is accompanied by a falling RoP (*Capital* 3). The qualitative analysis (yielding complex eigenvalues) confirms the cyclical nature of the system, while the simulations that we performed have shown that the system "stabilizes" in an approximately 50–year long cycle. The variable underneath all these phenomena is the long–run falling tendency in the RoP, a result consistent with the empirical regularity of long cycles (Korotayev and Tsirel 2010; Kleinknecht 2016; Shaikh 2016; Tsoulfidis and Tsaliki 2019).

Our growth and cycle model is in sharp contrast to Harrod's model in that the growth rate of surplus value defined as $g_C - g_V = \hat{e}$ becomes the key variable that regulates the turbulent but cyclical dynamics of capitalism. In particular, the growth rate of surplus value impacts both forces of production (investment, technology) and relations of production (employment, wages, class struggle). By contrast, the analysis of the warranted path in the Harrodian and Keynesian tradition defined as $g_C = g_V$, that is, the growth rate of capital is equal to the growth rate of employment (and by extension to the exogenous given natural growth rate) attributes the "knife–edge" problem to investors' decisions and to the exogenously determined "animal spirits." In our model, the growth of the rate of surplus value plays the role of the regulator that restores equilibrium once departing from it. More importantly, our answer to the "knife–edge" problem is also quite different to the Solovian solution based on substitutability and aggregate production functions of dubious validity and residually determined technology.

Lastly, our model considers various factors that affect profitability and economic growth, introducing them as equivalent dynamical variables and although it appears similar to the tradition of endogenous growth models, the conceptualization of the variables and their interaction with profitability and growth is generic and it is based on an overall systemic approach, rather than a specialized and unilateral treatment (such as the specific modelling of households' consumption or of firms' production profiles). In doing so, our model bridges an existing void in the literature that is the integration of a cycle and trend in a single overall model as it has been noted by Pasinetti (1960). The assumptions within a Harrodian or Solovian framework may be good enough to explain one "stylized fact" at a time, with the other needed to be exogenously introduced; this explains, to some extent, the poor explanatory power of these models over the years. In our model, the fusion of the key variables of capitalist growth and development leads inescapably to cyclical growth and to a simple but realistic explanation of the Harrodian "knife–edge" problem. Furthermore, the proposed model gives results which are robust to reasonable changes in parameters and maintains its qualitative features with respect to the trajectories of our variables.

References

Alexiou, Constantinos, and Persefoni Tsaliki. 2009. Unemployment revisited: Empirical evidence from 20 OECD countries. *Contributions to Political Economy* 28 (1): 23–34.

Armin, von Rudiger, and Jose Barrales. 2015. Demand–driven Goodwin cycles with Kaldorian and Kaleckian features. *Review of Keynesian Economics* 3 (3): 351–373.

Bellais, Renaud. 2004. Post Keynesian theory, technology policy, and long–term growth. *Journal of Post Keynesian Economics* 26 (3): 419–440.

Bergeaud, Antonin, Gilbert Cette, and Rémy Lecat. 2015. GDP per capita in advanced countries over the 20th century. Banque de France. Working Paper No. 549.

Blatt, John M. 2019. *Dynamic Economic Systems: A Post Keynesian Approach*. New York: Routledge.

Boldrin, Michele, Kazuo Nishimura, Tadashi Shigoka, and Makoto Yano. 2001. Chaotic equilibrium dynamics in endogenous growth models. *Journal of Economic Theory* 96 (1–2): 97–132.

Botwinick, Howard. 1993. *Persistent Inequalities: Wage Differentials under Capitalist Competition*. Princeton: Princeton University Press.

Chatzarakis, Nikolaos, and Persefoni Tsaliki. 2021. The dynamics of capital accumulation in Marx and Solow. *Structural Change and Economic Dynamics* 57: 148–158.

Darity, William Jr. 2009. More cobwebs? Robert Solow, uncertainty, and the theory of distribution. *History of Political Economy* 41 (1): 149–160.

Desai, Meghnad. 1973. Growth cycles and inflation in a model of the class struggle. *Journal of Economic Theory* 6 (6): 527–545.

Domar, Evsey D. 1948. The problem of capital accumulation. *The American Economic Review* 38 (5): 777–794.

Fatás–Villafranca, Francisco, Gloria Jarne, and Julio Sánchez–Chóliz. 2012. Innovation, cycles and growth. *Journal of Evolutionary Economics* 22 (2): 207–233.

Flaschel, Peter. 2008. *The Macrodynamics of Capitalism: Elements for a Synthesis of Marx, Keynes and Schumpeter*. New York: Springer.

Flaschel, Peter, Reiner Franke and Willi Semmler. 2007. Kaleckian investment and employment cycles in postwar industrialized economies. In *Mathematical Economics and the Dynamics of Capitalism: Goodwin's Legacy Continued*, eds. Flaschel, Peter, and Michael Landesmann, 35–65. London: Routledge.

Galor, Oded. 2011. *Unified Growth Theory*. Princeton: Princeton University Press.

Glombowski, Jörg. 1983. A Marxian model of long–run capitalist development. *Zeitschrift für Nationalökonomie* 43 (4): 363–382.

Glombowski, Jörg, and Michael Krüger. 1987. Generalizations of Goodwin's growth cycle model. In *Economic Evolution and Structural Adjustment*, eds. Batten, David, John L. Casti, and Börge Johansson, 260–290. Springer: Berlin.

Goodwin, Richard M. 1967. A growth cycle. In *Socialism, Capitalism and Economic Growth*, ed. Feinstein, Charles H., 54–58. Cambridge: Cambridge University Press.

———. 1990. *Chaotic Economic Dynamics*. Oxford.

Gordon, David M. 1980. Stages of Accumulation and Long Economic Cycles. In *Processes of the World System*, eds. Hopkins, Terence K., and Immanuel Wallerstein, 9–45. Beverly Hills: Sage.

Greiner, Alfred, and Willi Semmler. 1996. Saddle path stability, fluctuations, and indeterminacy in economic growth. *Studies in Nonlinear Dynamics and Econometrics* 1 (2): 105–118.

Grossman, Henryk. 1929. *The Law of Accumulation and Breakdown of the Capitalist System*. London: Pluto Press. [1992].

Harrod, Roy F. 1939. An essay in dynamic theory. *The Economic Journal* 49 (193), 14–33.

Jarsulic, Marc. 1991. Endogenous credit and endogenous business cycles. In *Nicholas Kaldor and Mainstream Economics: Confrontation or Convergence?*, eds. Nell, Edward J., and Willi Semmler, 395–407. London: Macmillan.

Kaldor, Nicholas. 1957. A model of economic growth. *The Economic Journal* 67 (268): 591–624.

Kalecki, Michał. 1954. *Theory of Economic Dynamics*. London: George Allen and Unwin.

Katrakilidis, Constantinos, and Persefoni Tsaliki. 2008. Competing theories of unemployment and economic policies: Evidence from the US, Swedish and German Economies. *The Indian Economic Journal* 56 (3): 90–108.

Kleinknecht, Alfred. 2016. *Innovation Patterns in Crisis and Prosperity: Schumpeter's Long Cycle Reconsidered*. London: Springer.

Kondratiev, Nikolai, D. 1926. Long cycles of economic conjuncture. In *The Works of Nikolai D. Kondratiev*, eds. Makasheva, Natali, Warren J. Samuels, and Vincent Barnett, trans. Stephen S. Wilson, 25–63. London: Pickering & Chatto.

Korotayev, Andrey V., and Sergey V. Tsirel. 2010. A spectral analysis of world GDP dynamics: Kondratieff waves, Kuznets swings, Juglar and Kitchin cycles in global economic development, and the 2008–2009 economic crisis. *Structure and Dynamics* 4 (1): 3–57.

Korpinen, Pekka. 1987. A monetary model of long cycles. In *The Long–Wave Debate*, ed. Vasko, Tibor, 333–341. Berlin: Springer.

Li, Yi, and James S. Muldowney. 1993. On Bendixson's Criterion. *Journal of Differential Equations* 106 (1): 27–39.

Luxemburg, Rosa. 1913. *The Accumulation of Capital*. New York: Routledge. [2015].

Marx, Karl. 1867. *Capital* (Vol. 1). New York: International Publishers. [1977].

———. 1885. *Capital* (Vol. 2). New York: International Publishers. [1977].

——— .1894. *Capital* (Vol. 3). New York: International Publishers. [1977].

Parente, Stephen. 2001. The failure of endogenous growth. *Knowledge, Technology and Policy* 13 (4): 49–58.

Pasinetti, Luigi L. 1960. Cyclical fluctuations and economic growth. *Oxford Economic Papers* 12 (2): 215–241.

Phelps, Edmund. 1961. The golden rule of accumulation: A fable for growth men. *The American Economic Review* 51 (4): 638–643.

Robinson, Joan. 1965. *Essays in the Theory of Economic Growth*. London: Macmillan.

Sasaki, Hiroaki. 2013. Cyclical growth in a Goodwin–Kalecki–Marx model. *Journal of Economics* 108 (2): 145–171.

Sato, Yoshikazu. 1985. Marx–Goodwin growth cycles in a two–sector economy. *Journal of Economics* 45 (1): 21–34.

Schumpeter, Joseph A. 1939. *Business Cycles. A Theoretical, Historical and Statistical Analysis of the Capitalist Process*. New York: McGraw–Hill.

———. 1942. *Capitalism, Socialism and Democracy*. New York: Routledge. [2010]

Semmler, Willi. 1984. Marx and Schumpeter on competition, transient surplus profit and technical change. *Economie Appliquée* 37 (3–4): 419–455.

Setterfield, Mark. 2021. "Whatever Happened to the 'Goodwin Pattern'? Profit Squeeze Dynamics in the Modern American Labour Market." *Review of Political Economy*. DOI: 10.1080/09538259.2021.1921357

Shaikh, Anwar. 1989. Accumulation, finance, and effective demand in Marx, Keynes, and Kalecki. In *Financial Dynamics and Business Cycles: New Perspectives*, ed. Willi Semmler, 65–86. New York: M.E. Sharpe Inc.

———. 1990. Organic composition of capital. In *Marxian Economics*, eds. Eatwell, John, Murray Milgate, and Paul Newman, 304–309. London: Palgrave Macmillan.

———. 1992. The falling rate of profit as the cause of long waves: Theory and empirical evidence. In *New Findings in Long–Wave Research*, eds. Kleinknecht, Alfred, Ernest Mandel, and Immanuel Wallerstein, 174–202. London: Palgrave Macmillan.

———. 2016. *Capitalism: Competition, Conflict, Crises*. Oxford: Oxford University Press.

Solow, Robert M. 1957. Technical change and the aggregate production function. *The Review of Economics and Statistics* 39 (3): 312–320.

———. 1960. Investment and technical progress. *Mathematical Methods in the Social Sciences* 1: 48–93.

Trigg, Andrew B. 2006. *Marxian Reproduction Schema: Money and Aggregate Demand in a Capitalist Economy*. New York: Routledge.

Tsaliki, Persefoni. 2006. Marx on entrepreneurship: A note, *International Review of Economics* 53 (4): 592–602.

———. 2009. Economic development and unemployment: do they connect?. *International Journal of Social Economics* 36 (7): 773–781.

Tsoulfidis, Lefteris. 2017. Growth Accounting of the Value Composition of Capital and the Rate of Profit in the U.S. Economy: A Note Stimulated by Zarembka's Findings. *Review of Radical Political Economics* 49 (2): 303–312.

———. and Dimitris Paitaridis. 2019. Capital intensity, unproductive activities and the Great Recession in the US economy. *Cambridge Journal of Economics* 43 (3): 623–647.

———, and Aris Papageorgiou. 2019. The recurrence of long cycles: Theories, stylized facts and figures. *World Review of Political Economy* 10 (4): 415–448.

———, and Persefoni Tsaliki. 2014. Unproductive labor, capital accumulation and profitability crisis in the Greek economy. *International Review of Applied Economics* 28 (5): 562–585.

———, and Persefoni Tsaliki. 2019. *Classical Political Economics and Modern Capitalism: Theories of Value, Competition, Trade and Long Cycles*. New York: Springer.

Verhulst, Ferdinand. 2000. *Nonlinear Differential Equations and Dynamical Systems*. Berlin: Springer–Verlag.

Appendix A

OCC: organic composition of capital
RoP: rate of profit
C: constant capital stock
V: variable capital
S: surplus value
γ: proxy to OCC
π: rate of profit on circulating capital or profit margin on cost
φ: the ratio of stocks–over–flows
r: rate of profit, RoP
$e = {}^{S}/_{V}$: rate of surplus value
$\hat{\gamma}$: growth rate of OCC
$\hat{\pi}$: growth rate of rate of profit on cost or profit margin
$\hat{r}$: growth rate of RoP
$\hat{e}$: growth rate of rate of surplus value
$\hat{C}$: growth rate of constant capital,
$\hat{V}$: growth rate of variable capital
δ: devaluation
s_C: share of surplus value that potentially can be invested in capital
$\widehat{s_C}$: growth rate of s_C
s_V: share of surplus value that potentially can be invested in variable capital
τ: rate of technological change
a_1 and a_2 are relatively small positive parameters that help us to scale r appropriately and may be seen as the elasticities of the RoP with respect to technological and distributional parameters, respectively
b_1 is the intensity of introduction of innovations in the economy
b_2 is the level of the devaluation rate where no technological 'jump' occurs and productivity rises steadily
g_n: 'natural' growth rate or growth rate of employment
n: growth rate of population
g_w: threshold growth rate
λ_i: real eigenvalue
$\vec{v}_i$: eigenvector
* : denotes equilibrium
$\dot{}$: denotes derivative with respect to time
$\widehat{}$: denotes rate of change

Appendix B

B1: Derivation of the relations for $\hat{\gamma}$ and $\hat{\pi}$ (Eqs. 5)

As in Chatzarakis and Tsaliki (2021), we consider the rate of change of the constant capital to be equal to the investment in constant capital minus the depreciation:

$$g_C = \frac{\dot{C}}{C} = s_C \frac{\pi}{\gamma} - \delta$$

and the rate of change of the variable capital to be equal to the investment in variable capital (hiring or firing of laborers) plus the rate of technological change:

$$g_V = \frac{\dot{V}}{V} = s_V \frac{\pi}{1-\gamma} + \tau$$

From the definition of γ, we have:

$$\hat{\gamma} = \frac{\dot{\gamma}}{\gamma} = \frac{\left({}^{C}\!/_{C+V}\right)^{\cdot}}{{}^{C}\!/_{C+V}} = \frac{\dot{C}}{C} - \frac{\dot{C}+\dot{V}}{C+V} = \frac{\dot{C}}{C}\left(1 - \frac{C}{C+V}\right) - \frac{\dot{V}}{V}\frac{V}{C+V}$$

Since $1 - C(C+V)^{-1} = V(C+V)^{-1} = 1-\gamma$,

$$\hat{\gamma} = \frac{\dot{\gamma}}{\gamma} = (1-\gamma)(g_C - g_V)$$

and substituting:

$$\dot{\gamma} = s_C(1-\gamma)\pi - s_V\gamma\,\pi - \gamma(1-\gamma)(\delta+\tau)$$

From the definition of π, we have:

$$\hat{\pi} = \frac{\dot{\pi}}{\pi} = \frac{\left({}^{S}\!/_{C+V}\right)^{\cdot}}{{}^{S}\!/_{C+V}} = \frac{\dot{S}}{S} - \frac{\dot{C}+\dot{V}}{C+V} = \frac{(eV)^{\cdot}}{eV} - \frac{\dot{C}}{C}\frac{C}{C+V} - \frac{\dot{V}}{V}\frac{V}{C+V}$$
$$= \hat{e} - \frac{\dot{C}}{C}\frac{C}{C+V} + \frac{\dot{V}}{V}\left(1 - \frac{V}{C+V}\right)$$

Similarly:

$$\dot{\pi} = \hat{e}\pi - s_C\pi^2 + s_V\frac{\gamma}{1-\gamma}\pi^2 + \gamma\,\pi(\delta+\tau)$$

B2: Derivation of the relation for $\hat{r}$ (Eq. 8)

Given equations (5), and the definition of r, we have:

$$\hat{r} = \frac{\dot{r}}{r} = \frac{\left({}^{\pi}\!/_{\gamma}\right)^{\cdot}}{{}^{\pi}\!/_{\gamma}} = \frac{\dot{\pi}}{\pi} - \frac{\dot{\gamma}}{\gamma}$$

and substituting:

$$\hat{r} = \frac{\dot{r}}{r} = \hat{e} - s_C\pi + s_V\frac{\gamma}{1-\gamma}\pi + \gamma(\delta+\tau) - s_C\frac{1-\gamma}{\gamma}\pi + s_V\pi + (1-\gamma)(\delta+\tau)$$
$$= \hat{e} - s_C\frac{\pi}{\gamma} + s_V\frac{\pi}{\gamma} + \delta + \tau$$

Consequently:

$$\dot{r} = -(s_C - s_V)r^2 + (\delta + \tau + \hat{e})r$$

From *Capital* I, we know that the rate of surplus value changes inversely to the changes of the employment rate, hence $\hat{e} \approx -s_V$ (see Glombowski 1983). Using the direct analogy ($\hat{e} = -s_V$), we need to counterbalance for any deviations. In order to do so and to ensure that solutions are economically meaningful ($0 < r < 1$), we multiply with two positive and smaller–than–unity parameters, a_1 and a_2, hence:

$$\dot{r} = -a_1(s_C - s_V)r^2 + a_2(\delta + \tau - s_V)r$$

B3: Derivation of the relation for $\dot{s}_V$ (Eq. 9)

s_V is perceived as the change in employment, ϵ. We can assume that it captures non–linear deviations of ϵ from its equilibrium value ϵ^*, as computed by Goodwin (1967). A simple form of these non–linear deviations is the logarithmic change, hence:

$$s_V = \ln \epsilon - \ln \epsilon^* = \ln\left(\frac{\epsilon}{\epsilon^*}\right)$$

or in an integral form:

$$s_V = \int_{\epsilon^*}^{\epsilon} \frac{\mathrm{d}\epsilon}{\epsilon}$$

This form is chosen because its linear approximation is merely the linear deviation $s_V \cong {}^{\epsilon - \epsilon^*}/_{\epsilon^*}$.
From Goodwin (1967), we know that:

$$\frac{\dot{\epsilon}}{\epsilon} = s_C \frac{1-\omega}{\sigma} - n + \tau + \delta$$

where ω the wage share, $1 - \omega$ the profit share and σ the capital–output ratio; obviously:

$$\frac{1-\omega}{\sigma} = \frac{S}{Y}\frac{Y}{C} = \frac{S}{C} = r$$

where $Y = S + V$; hence:

$$\frac{\dot{\epsilon}}{\epsilon} = s_C r - n + \tau + \delta$$

Differentiating s_V with respect to time:

$$\dot{s}_V = \frac{\mathrm{d}}{dt}\ln\left(\frac{\epsilon}{\epsilon^*}\right) = \frac{\dot{\epsilon}}{\epsilon^*}\frac{\epsilon^*}{\epsilon} = \frac{\dot{\epsilon}}{\epsilon}$$

so that:

$$\dot{s}_V = s_C r - n + \tau + \delta$$

SHORT BIOS

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