# Introduction, analysis and asymptotic behavior of a multi-level manpower planning model in a continuous time setting under potential department contraction

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**Abstract**

A mathematical model in a multi-level manpower planning setting is developed and analyzed incorporating the divisions of an organization's personnel into several homogeneous groups. The proposed framework builds upon recent research to develop, via the continuous time scale, a departmental model encompassing employees flows within departments (intra-departmental transitions), as well as transfers among departments (inter-departmental transitions). Management-wise, this is a common practice under certain conditions as in restructuring and rightsizing ventures both in private industries and in the public sector. After establishing the baseline differential equations of the system, an investigation follows regarding the system's limiting behavior integrating the concept of department contraction which can be seen as an indirect and mild control that can be exercised on the manpower system. It is proved that under a set of conditions, the system's limiting structure exists and is specified. The aforementioned asymptotic analysis can be utilized to predict, in the long run, future stock structures of departments and thus, in the case where these structures diverge from the desired ones, identify departments that could be considered as candidates for probable rightsizing or mergers. The paper concludes with a numerical illustration and some concluding remarks.

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# 1 Introduction

Mathematical human resource planning and especially manpower planning mainly refers to a cluster of problems commonly known as staffing or personnel planning problems, in a continuous effort to obtain insight into how change takes place in an organization comprising people and infrastructure. A manpower system is usually made up from people with common goals within a company or organization, along with the inherent uncertainties existing in human behavior. The evolving process of any manpower system often lends itself to stochastic modeling in the design, development and further analysis of such a system (Babu and Rao 2017; Udom 2013). In this general modeling framework, the personnel of an organization (manpower system) is stratified into hierarchical (or non-hierarchical) structures and the distribution of the employees in various states or subgroups (based on grades, classes, length of stay, skills etc) is usually represented by state vectors (stock vectors). These groups are typically mutually exclusive and exhaustive. Then, these state vectors are manipulated in various ways, following imposed assumptions and constraints or even purposeful interventions, in order to achieve miscellaneous goals. Such goals might regard estimating projections of future personnel structures and investigating their limiting behavior, assigning properly qualified persons required to a system's state, as well as allocating available personnel on duties and shifts. Other goals may consider the detection of appropriate courses of action and handling optimization objectives regarding efficient utilization of labor under reasonable costs or rewards, the evaluation of current manpower structures and their potential in reaching desired structural configurations using various control aspects, the experimentation using sensitivity analysis of existing or future availability of human resources and so on. In this respect, the Markov approach to manpower planning suggests using stocks and flows inwards (recruitment), within (internal transfers) or outwards (wastage) of the system, realized by transition probabilities (Bartholomew, Forbes, and McClean 1991) to model employee mobility and, in this sense, reflecting the stochastic behavior of personnel transfers throughout time.

There is a rich body of literature on Markov manpower planning dealing with the above inquiries. By focusing on the time scale, the bulk of them utilizes Markov (or semi Markov) theory in discrete time. In these cases, the states of a system with $k$ grades are denoted by $S_1, S_2, ..., S_k$ and the personnel stocks at discrete time points $t$, are denoted by row (or column) vectors $\mathbf{N}(t)$, where $t = 0, 1, 2, ...$ Some characteristic examples of research regard maintainability, attainability, desirability and promotion steadiness, roster quality or evaluation of staffing policies, optimal control by recruitment or alternative approaches, asymptotic and limiting behavior and can be found in early studies such as, Bartholomew (1982), Bartholomew, Forbes, and McClean (1991), Vassiliou (1998), Ugwuowo and McClean (2000), Georgiou and Tsantas (2002), McClean, Papadopoulou, and Tsaklidis (2004), Nilakantan and Raghavendra (2005), Wang (2005), Guerry (2008) and Dimitriou and Tsantas (2009). More recent examples are: De Feyter and Guerry (2011), Komarudin et al. (2013), Udom (2013, 2014), Guerry (2014), Komarudin et al. (2015), Nilakantan (2015), Komarudin et al. (2016), De Feyter, Guerry, and Komarudin (2017), Kyritsis and Papadopoulou (2017) and Jaillet, Loke, and Sim (2018). In addition, much research has been done also in the direction of studying the limiting behavior of Markov manpower systems. To consult some of these works both in discrete and continuous time scale on this domain, see, for example, Bartholomew (1982), Vassiliou (1982), Gerontidis (1990a, 1990b), Vassiliou and Georgiou (1990), Yadavalli, Natarajan, and Udayabhaskaran (2002), Papadopoulou and Vassiliou (1994, 1999), Tsantas (2001), Guerry (2008), Dimitriou and Tsantas (2010), Guerry and De Feyter (2011), Dimitriou and Tsantas (2012), Papadopoulou and Vassiliou (2014) and Vassiliou (2014). Analogous Markov models have been also formulated in continuous time. A good account of the early relevant literature can be found in McClean, Montgomery, and Ugwuowo (1998). Some of the most recent contributions in the continuous time Markov manpower modeling can be found in Vasiliadis and Tsaklidis (2009), Vasiliadis (2014) and Babu and Rao (2017). Although these models are mathematically more sophisticated than the discrete time ones, they inevitably carry a higher degree of complexity with the expected compensation of a more realistic approach by allowing a rigorous monitoring of the system at hand. In this sense, there exist two main approaches concerning the application of continuous time manpower models.

a. The first concerns the systems in which the differential equation(s) that govern its evolvement can be solved analytically. In this case, various methods are used in order to estimate its parameters (McClean, Montgomery, and Ugwuowo 1998; Inamura 2006) mainly from discrete time observations (data collection) over a time window. Then, the continuous time model is applied as is for the consequent analysis of the system.

b. The second approach deals with the systems in which the complexity of the differential equations or the assumption of non-homogeneity of the models' parameters make it difficult to apply in practice. In such cases, researchers resort to discrete time (numerical) approximations to make the model applicable (see for example Janssen and Manca (2002) and Kowalczuk (1993))

In the following stages of the formulation of our model, we will assume time homogeneity of the parameters, a fact that enables us to develop a continuous time Markov model which would be applied in practice on the grounds of the first approach.

In this paper we adopt the pragmatic realization of a manpower establishment in which the personnel are divided into departments according to skills, duties, processes or even particular flow of product or services. In this respect, we begin with a multi-level manpower planning model (see for example, Guerry and De Feyter (2012)) that essentially implies a kind of personnel non homogeneity. Under certain conditions, implied either by the firm conditions or by the career prospective and aspirations of an employee, a person residing in one department might consider his/her options of transferring to another department or even to a different branch. There are circumstances though, such as periods of external economic or internal functional turbulence that might render these kind of employee transfers among similar (or dissimilar) departments, imperative. Common management practices are those of rightsizing or mergers, resulting into workforce mobility in the form of either reduction (attrition) or departmental transfers (internal horizontal mobility). Similar strategies, albeit less flexible, can be also found in general public services where in periods of economic recession, measures of spending cuts in central government or even personnel reallocation are commanding in order to achieve economies of scale or scope (Dimitriou, Georgiou, and Tsantas 2013). In addition, it may be also argued that interdepartmental transitions result into personnel groups with a broad knowledge of organizational functions and management culture, a fact that might address imbalances between departments with respect to quantity as well as staff qualifications (Guerry and De Feyter 2012). The above ideas have triggered, in the last decade, a series of Markov manpower modeling approaches that on one hand take advantage of the notion of multi-level divisions into specialized departmental levels (in turn comprised by departmental grades) and on the other hand permit interdepartmental transitions caused either by external or internal causes (Ossai and Uche 2009; Guerry and De Feyter 2012; Dimitriou, Georgiou, and Tsantas 2013, 2015). It is worth noticing however, that although the above modeling efforts bring an interesting variety of modeling approaches within the general framework of Markov manpower systems, they all adopt a discrete time scale to accommodate the events that take place during the realization of the mobility process (i.e. to reflect the events of recruitment, internal transitions and attrition in various forms).

The current paper builds upon the discrete time manpower model introduced in Dimitriou, Georgiou, and Tsantas (2015) to capture inherent characteristics of the mobility behavior in continuous time scale. In this respect, this paper is considered as part 2 of this earlier work and apart from the scale turning into a continuous setting it also brings along some additional ideas such as a systematic investigation of department contraction (or expansion) as well as the necessary mathematical manipulation to reveal its asymptotic behaviour and analysis. Since, to our knowledge there have not been relevant research attempts in continuous time, one of the goals of the present work is to formulate a continuous time multi-level Markov model that would assist in gaining a continual insight of the internal and, more specifically, departmental mobility of a system. In this context, we establish a functional form of the so-called Continuous Time Multilevel Homogeneous Manpower System (CTMHMS) that could project not only the mobility of employees inside the grades of each department, but also the relevant mobility among the grades of different departments. This continuous-time modelling proposal, incorporates in the CTMHMS the next aspect: the analysis of the management's possible attempt for the contraction of some departments, reflecting its intention to eventually shut down some departments. Although shutting down a department seems a rather radical downsizing approach, this contraction concept offers a mild and indirect control tool that in parallel with the study of the system's limiting behavior (accomplished in the second part of this paper) reveals potential candidate departments for rightsizing or merging. As already mentioned, a third goal is the investigation of the asymptotic behavior of the CTMHMS under particular assumptions. In addition, it is proved that under certain conditions, a structural reform of the manpower system (downsizing, merging of candidate departments) can be achieved fast enough if required (exponential rate).

The rest of the paper is structured as follows: In Section 2 we demonstrate the development of the model. We introduce the parameters that regulate the flows of employees within and amongst departments and provide the corresponding differential equations that govern these flows. The section continues with the provision of the equations that determine the system's expected population structure at any time $t \geq 0$, and concludes with the description of the notion of department contraction as a means of moderate control. Section 3 is about the analysis of the liming behavior of the system under various conditions. The asymptotic behavior of the CTMHMS reveals the magnitudes that the expected personnel stocks tend to reach in the course of time and in this sense, the study of this behavior acts as a predictor of potential redundant departments. In this respect, it offers proposals of possible measures in pursuing more efficient organizational structures to achieve economies of scale or scope. Section 4 presents a numerical example that illustrates the theoretical results of the previous sections. Then, Section 5 follows comprising a summary of the current work and introducing some ideas for further research. Finally, the Appendix provides a synopsis of the concepts and results necessary for the study of the liming behavior of the system presented in Section 3.

# 2 Model Development

Assume that the human resources of a manpower system are divided into $D$ departments (on the basis of their specialized skills) and the employees of the $d$-th department, $d = 1, 2, ..., D$, are categorized into $k_{(d)}$ homogeneous classes (states) represented by the sets $S_{(d)} = \{1_{(d)}, 2_{(d)}, \ldots, k_{(d)}\}$. We accept that the aforementioned states are mutually exclusive and exhaustive. Taking into account that the classes of any given department $\mu$ may be hierarchical, we denote by $S_{(\mu)} = \{1_{(\mu)}, 2_{(\mu)}, \ldots, k_{(\mu)}\}$ the set consisting of the $k_{(\mu)}$ system's grades, where $k_{(\mu)}$ denotes the top grade of the hierarchy. By considering time as a continuous parameter, the (row) vector $\mathbf{N}(t) = [\mathbf{N}_{(1)}(t) \mid \mathbf{N}_{(2)}(t) \mid \ldots \mid \mathbf{N}_{(D)}(t)] = [N_{(1),1}(t) \;\; N_{(1),2}(t) \; \ldots \; N_{(1),k_{(1)}}(t) \mid \ldots \mid N_{(D),1}(t) \;\; N_{(D),2}(t) \; \ldots \; N_{(D),k_{(D)}}(t)]$ represents the expected system's state vector at time $t \geq 0$, with $N_{(d),s}(t)$ being the expected number of employees residing in class $s,\ s = 1, \ldots, k_{(d)}$ of department $d$, $d = 1, \ldots, D$ at time $t$.

Let also that the personnel transfers of the system entail both *intra*-mobility i.e. movements of employees among the classes of a single department, and *inter*-mobility i.e. movements among the states of different departments. In addition, the system is considered open i.e. there exist recruitment and wastage flows.

## 2.1 Inter-Mobility rates

We begin by providing the *inter*-mobility configuration of the system. Suppose that $q_{(db)}$ $d, b = 1, 2, ..., D$, $d \neq b$, denotes the time homogeneous intensity (rate) of a movement from department $d$ to $b$. Furthermore, $\mathbf{P}_{(db)} = \{p_{(db),ij}\}$, $d, b = 1, 2, ..., D$, $d \neq b$, $i \in S_{(d)}$, $j \in S_{(b)}$ stands for a $k_{(d)} \times k_{(b)}$ row stochastic probability matrix with entries the conditional probabilities provided below:

$$p_{(db),ij} = \text{prob}\{\text{an employee will be transferred from state } i \in S_{(d)} \text{ to state } j \in S_{(b)}, \textit{ given} \text{ that movements of employees from department } d \text{ to } b \text{ (i.e. } q_{(db)} \neq 0\text{) will take place in } (t, t+\delta t),\ t \geq 0\}.$$

As a consequence, the products $q_{(db)}p_{(db),ij}$ $\forall d, b = 1, 2, ..., D$, $d \neq b$, $i \in S_{(d)}$, $j \in S_{(b)}$ represent the homogeneous transition rates from state $i \in S_{(d)}$ to state $j \in S_{(b)}$.

*Remark* 2.1. The above products have their origin to the next logical framework: If we define $q_{(d),ii}$ as the sum of all rates that correspond to transitions from state $i$ of department $d$ (see Equation (2) below), then $\frac{q_{(db)}}{q_{(d),ii}}$ is the probability of the embedded discrete-time Markov (jump) chain that there will occur movements from department $d$ to $b$ (in the next transition). Hence, the (unconditional) probability of an employee to be transferred in his/her next transition (jump) from state $i \in S_{(d)}$ to state $j \in S_{(b)}$ would be equal to $\frac{q_{(db)}}{q_{(d),ii}} \cdot p_{(db),ij}$. As a result, the corresponding rates for these transitions ought to be equal to $q_{(db)}p_{(db),ij}$.

## 2.2 Intra-Mobility and wastage rates

Consider now the *intra*-mobility rates within the grades of a single department. Assume that $q_{(dd),ij}$ $i, j \in S_{(d)}$, $d = 1, 2, ..., D$, stands for the transition rate of an employee residing in class $i \in S_{(d)}$ of department $d$ to move in class $j \in S_{(d)}$, $i \neq j$ of the same department. Furthermore, we accept that departures from the internal classes of $d$ towards the external environment can take place, with $w_{(d),i}$ being the rate of an individual occupying $i \in S_{(d)}$, to abandon $d$ due to reasons, such as retirement, job hoping etc. Let $\mathbf{w}_{(d)} = [w_{(d),1} \ \ldots \ w_{(d),k_{(d)}}]$ denote the vector gathering these wastage rates of $d$.

In the sequel, we define:

$$q_{(d)} = \sum_{b=1, b\neq d}^{D} q_{(db)} \tag{1}$$

$$q_{(d),ii} = q_{(d)} + \sum_{s=1, s\neq i}^{k_{(d)}} q_{(dd),is} + w_{(d),i} \tag{2}$$

As a consequence (Kulkarni 1995):

$$\begin{aligned}&\text{prob}\{\text{an employee occupying } i \in S_{(d)} \text{ at time } t \text{ remains in } i \text{ in } (t, t+\delta t)\}\\ &= 1 - q_{(d),ii}\delta t + o(\delta t)\end{aligned}$$

Let also:

$$\mathbf{Q}_{(dd)} := \begin{pmatrix} -q_{(d),11} & q_{(dd),12} & \cdots & q_{(dd),1k_{(d)}} \\ q_{(dd),21} & -q_{(d),22} & \cdots & q_{(dd),2k_{(d)}} \\ \vdots & & \ddots & \vdots \\ q_{(dd),k_{(d)}1} & q_{(dd),k_{(d)}2} & \cdots & -q_{(d),k_{(d)}k_{(d)}} \end{pmatrix}$$

be the matrix consisting of the intra-mobility intensities having all its row sums negative (since there are also transitions, and thus non-zero intensities, from department $d$ towards other departments). In particular, the sum of its $i \in S_{(d)}$ row, according to (2), equals to $-q_{(d)} - w_{(d),i}$.

## 2.3 Recruitment rates

Assume that $T_{(d)}(t)$ expresses the known continuous function of the total expected number of employees being in state $d$, at time $t \in \mathbb{R}^+$, considered to be determined by the management of the system. Only when department $d$ is *expanded* in $(t, t+\delta t)$, we define $M_{(d)}(t) = dT_{(d)}(t)/dt \geq 0$ to be the rate of increase (marginal increase) of the size of $d$. In all other cases $M_{(d)}(t)$ is considered zero. Suppose also that $N_{(d),w}(t+\delta t)$ denotes the number of employees that abandon $d$ for the external environment in $(t, t+\delta t)$. Then, in $(t, t+\delta t)$, due to the fact that

$$\begin{aligned}&\text{prob}\{\text{a person residing state } i \in S_{(d)} \text{ in } t \text{ leaves department } d \text{ for the}\\ &\text{external environment in } (t, t+\delta t)\}\\ &= w_{(d),i}\delta t + o(\delta t)\end{aligned}$$

we get

$$\begin{aligned} E[N_{(d),w}(t+\delta t)] &= E[E[N_{(d),w}(t+\delta t)|\mathbf{N}_{(d)}(t)]] \\ &= \mathbf{N}_{(d)}(t)\mathbf{w'}_{(d)}\delta t + o(\delta t) \end{aligned}$$

which means that a number of $\mathbf{N}_{(d)}(t)\mathbf{w'}_{(d)}\delta t + o(\delta t)$ employees are expected to leave $d$ for the external environment in $(t, t+dt)$. With analogous reasoning, it can be shown that $T_{(d)}(t)\sum_{m=1,m\neq d}^{D} q_{(dm)}\delta t + o(\delta t)$ employees are expected to leave $d$ moving towards other departments and $\sum_{b=1,b\neq d}^{D} T_{(b)}(t)q_{(bd)}\delta t + o(\delta t)$ individuals are expected to arrive at $d$ from other departments in the same time interval. In addition, the fact that there should be ample vacancies for the employees who move among departments, calls for the establishment of a constraint like the following for every interval $(t, t+dt)$:

$$\begin{aligned} &\underbrace{M_{(d)}(t)\delta t + o(\delta t)}_{\substack{\text{new posts in} \\ \text{department } d}} + \underbrace{T_{(d)}(t)\sum_{\substack{m=1 \\ m\neq d}}^{D} q_{(dm)}\delta t + o(\delta t)}_{\substack{\text{empty posts} \\ \text{in department } d \\ \text{due to transfers} \\ \text{of employees to} \\ \text{other departments } \neq d}} + \underbrace{\mathbf{N}_{(d)}(t)\mathbf{w'}_{(d)}\delta t + o(\delta t)}_{\substack{\text{empty posts} \\ \text{in department } d \\ \text{due to wastage}}} \\ &\geq \underbrace{\sum_{\substack{b=1 \\ b\neq d}}^{D} T_{(b)}(t)q_{(bd)}\delta t + o(\delta t)}_{\substack{\text{expected number} \\ \text{of persons} \\ \text{transferred to } d \\ \text{from other departments}}} \end{aligned} \tag{3}$$

By dividing (3) by $\delta t$ and letting $\delta t \to 0$ we get

$$M_{(d)}(t) + T_{(d)}(t)\sum_{\substack{m=1 \\ m\neq d}}^{D} q_{(dm)} + \mathbf{N}_{(d)}(t)\mathbf{w'}_{(d)} \geq \sum_{\substack{b=1 \\ b\neq d}}^{D} T_{(b)}(t)q_{(bd)} \tag{4}$$

Inequality (4) expresses the fact that the total number of available posts in $d$ (new or vacancies) in the time interval $(t, t+\delta t)$, should be no less than the number of individuals moved from other departments of the company towards $d$, in $(t, t+\delta t)$. We call the rate of the number of the posts that remain empty after inter-department movements *external recruitment rate* of department $d$, and we symbolize it by

$r_{(d)}(t) := M_{(d)}(t) + T_{(d)}(t) \sum_{m=1, m\neq d}^{D} q_{(dm)} + \mathbf{N}_{(d)}(t)\mathbf{w'}_{(d)} - \sum_{b=1, b\neq d}^{D} T_{(b)}(t) q_{(bd)}$. These positions are filled by recruitment inflow from the external environment. Let $\mathbf{p}_{(d)0} = \{p_{(d)0,i}\}$, $i \in S_{(d)}$ be the stochastic row vector of the input probabilities distribution in the various states of department $d$, $d = 1, ..., D$, of new members assigned to the above posts. More particularly its elements are equal to:

$$p_{(d)0,i} \;=\; \text{prob}\{\text{a recruit entering the system in class } i \in S_{(d)} \textit{ provided} \text{ that this recruit is going to cover one of the } r_{(d)}(t)\delta t + o(\delta t) \text{ remaining empty posts in } (t, t+\delta t),\ t \geq 0, \text{ in department } d\}.$$

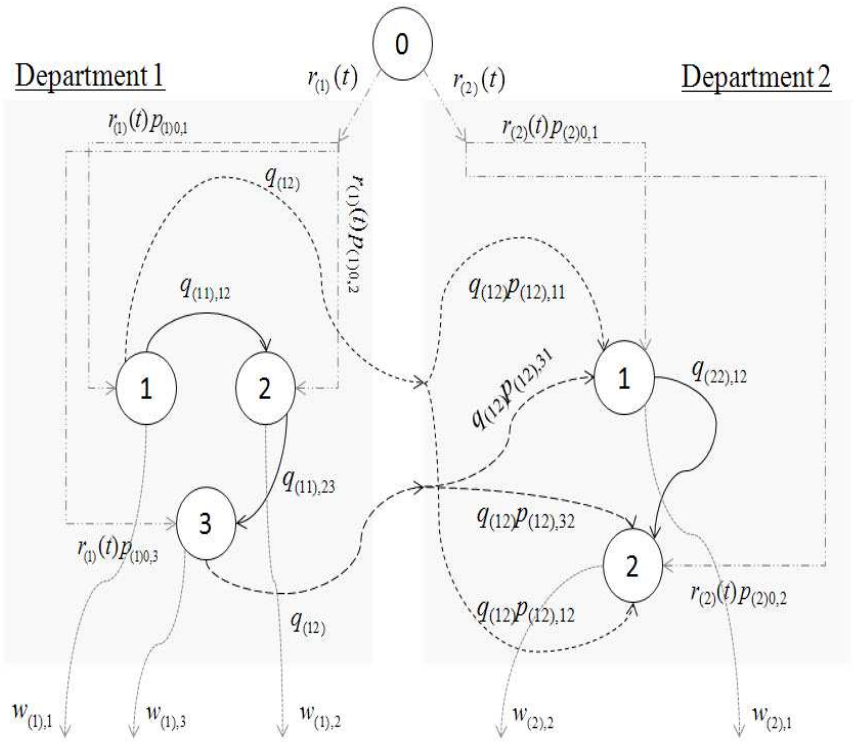


Figure 1: Rate diagram of a CTMHMS with $D = 2$ departments.

The aforesaid framework including:

- the inter-mobility rates of subsection 2.1,
- the intra-mobility and wastage rates of subsection 2.2,
- constraint (4) and
- the recruitment rates of subsection 2.3,

make up the basic parameters of the so-called *Continuous Time Multi-level Homogeneous Manpower System* (CTMHMS).

In order to explicate the utilization of these parameters, we depict in Figure 1 a rate diagram of a CTMHMS with $D = 2$ departments and $k_{(1)} = 3, k_{(2)} = 2$. For this system is assumed that

$$\mathbf{Q}_{(11)} = \begin{pmatrix} -q_{(1),11} & q_{(11),12} & 0 \\ 0 & -q_{(1),22} & q_{(11),23} \\ 0 & 0 & -q_{(1),33} \end{pmatrix}, \mathbf{Q}_{(22)} = \begin{pmatrix} -q_{(2),11} & q_{(22),12} \\ 0 & -q_{(2),22} \end{pmatrix},$$

$$\mathbf{P}_{(12)} = \begin{pmatrix} p_{(12),11} & p_{(12),12} \\ 0 & 0 \\ p_{(12),31} & p_{(12),32} \end{pmatrix}$$ and $q_{(21)} = 0$ (no transfers from department 2 to 1).

Via the parameters defined in the previous subsections, and by using conditional expectation arguments, the following system of differential equations describes the evolution the expected number of personnel in department $d$, $d = 1, ..., D$:

$$\frac{d\mathbf{N}_{(d)}(t)}{dt} = \mathbf{N}_{(d)}(t)\mathbf{Q}_{(dd)} + \sum_{\substack{b=1 \\ b\neq d}}^{D} \mathbf{N}_{(b)}(t) q_{(bd)} \mathbf{P}_{(bd)} + r_{(d)}(t)\mathbf{p}_{(d)0}, \quad d = 1, 2, ..., D \tag{5}$$

Let us now define:

$$\theta_{(d)}(t) := M_{(d)}(t) + T_{(d)}(t) \sum_{\substack{m=1 \\ m\neq d}}^{D} q_{(dm)} - \sum_{\substack{b=1 \\ b\neq d}}^{D} T_{(b)}(t) q_{(bd)} \tag{6}$$

and

$$\frac{d\mathbf{N}(t)}{dt} \quad := \quad \left[ \frac{d\mathbf{N}_{(1)}(t)}{dt} \middle| \frac{d\mathbf{N}_{(2)}(t)}{dt} \middle| \ldots \middle| \frac{d\mathbf{N}_{(D)}(t)}{dt} \right] \tag{7}$$

Then, the corresponding differential equations that govern the evolution of the entire system take the form:

$$
\begin{aligned}
\frac{d\mathbf{N}(t)}{dt} &= [\mathbf{N}_{(1)}(t) \mid \mathbf{N}_{(2)}(t) \mid \ldots \mid \mathbf{N}_{(D)}(t)] \\
&\quad \times \left( \begin{array}{c|c|c|c}
\mathbf{Q}_{(11)} & q_{(12)}\mathbf{P}_{(12)} & \cdots & q_{(1D)}\mathbf{P}_{(1D)} \\ \hline
q_{(21)}\mathbf{P}_{(21)} & & & q_{(2D)}\mathbf{P}_{(2D)} \\ \hline
\vdots & & \ddots & \vdots \\ \hline
q_{(D1)}\mathbf{P}_{(D1)} & q_{(D2)}\mathbf{P}_{(D2)} & \cdots & \mathbf{Q}_{(DD)}
\end{array} \right) \\
&\quad + \left[ r_{(1)}(t)\mathbf{p}_{(1)0} \mid r_{(2)}(t)\mathbf{p}_{(2)0} \mid \ldots \mid r_{(D)}(t)\mathbf{p}_{(D)0} \right] \\
&= [\mathbf{N}_{(1)}(t) \mid \mathbf{N}_{(2)}(t) \mid \ldots \mid \mathbf{N}_{(D)}(t)] \\
&\quad \times \left( \begin{array}{c|c|c}
\mathbf{Q}_{(11)} + \mathbf{w}'_{(1)}\mathbf{p}_{(1)0} & \cdots & q_{(1D)}\mathbf{P}_{(1D)} \\ \hline
q_{(21)}\mathbf{P}_{(21)} & & q_{(2D)}\mathbf{P}_{(2D)} \\ \hline
\vdots & \ddots & \vdots \\ \hline
q_{(D1)}\mathbf{P}_{(D1)} & \cdots & \mathbf{Q}_{(DD)} + \mathbf{w}'_{(D)}\mathbf{p}_{(D)0}
\end{array} \right) \\
&\quad + \left[ \theta_{(1)}(t)\mathbf{p}_{(1)0} \mid \theta_{(2)}(t)\mathbf{p}_{(2)0} \mid \ldots \mid \theta_{(D)}(t)\mathbf{p}_{(D)0} \right]
\end{aligned} \tag{8}
$$

In the sequel, let:

$$
\bar{\mathbf{Q}} := \left( \begin{array}{c|c|c}
\mathbf{Q}_{(11)} + \mathbf{w}'_{(1)}\mathbf{p}_{(1)0} & \cdots & q_{(1D)}\mathbf{P}_{(1D)} \\ \hline
q_{(21)}\mathbf{P}_{(21)} & & q_{(2D)}\mathbf{P}_{(2D)} \\ \hline
\vdots & \ddots & \vdots \\ \hline
q_{(D1)}\mathbf{P}_{(D1)} & \cdots & \mathbf{Q}_{(DD)} + \mathbf{w}'_{(D)}\mathbf{p}_{(D)0}
\end{array} \right)
$$

and

$$
\boldsymbol{\theta}(t) := \left[ \theta_{(1)}(t)\mathbf{p}_{(1)0} \mid \theta_{(2)}(t)\mathbf{p}_{(2)0} \mid \ldots \mid \theta_{(D)}(t)\mathbf{p}_{(D)0} \right].
$$

Note that $\bar{\mathbf{Q}}$ is an infinitesimal generator-rate matrix (its row sums equal to zero). With the aid of the above definitions, Equation (8) can be written as:

$$
\frac{d\mathbf{N}(t)}{dt} = \mathbf{N}(t)\bar{\mathbf{Q}} + \boldsymbol{\theta}(t) \tag{9}
$$

Suppose now that $\boldsymbol{\Phi}(t)$ is the transition probability matrix that corresponds to the continuous time Markov chain with infinitesimal generator $\bar{\mathbf{Q}}$. Then it is well known (Kulkarni 1995), that $\boldsymbol{\Phi}(t) = e^{\bar{\mathbf{Q}}t} = \sum_{\kappa=0}^{\infty} \frac{(\bar{\mathbf{Q}}t)^{\kappa}}{\kappa!}$. Moreover, from (9) we have that the expected structure $\mathbf{N}(t)$, $t \geq 0$, equals:

$$
\mathbf{N}(t) = \mathbf{N}(0)\boldsymbol{\Phi}(t) + \int_0^t \boldsymbol{\theta}(x)\boldsymbol{\Phi}(t-x)dx \tag{10}
$$

Expression (10) provides the actual structural relation of the CTMHMS's behavior for any $t \geq 0$.

It is worth noticing at this point that, the CTMHMS may constitute an alternative approach for modeling introductory training classes as in Dimitriou and Tsantas (2009, 2012) in continuous time. In the formulation of such an approach the training classes can be represented as the classes of a separate subgroup-department of the CTMHMS, and the preparation classes as the classes of another subgroup. In this way the CTMHMS can project both the tendency of employees to attend seminar courses so as to improve their career prospects, and the organizations' intention to create an inventory of skilled individuals for hiring.

## 2.4 Characterization of the departments' sizes

In this subsection we concentrate on how the (known) function of the expected size $T_{(d)}(t)$ of (each) department $d = 1, \ldots, D$ may change as time goes by, affecting in the sequel the evolution of the expected structure of the system $\mathbf{N}(t)$. Towards this direction we define $\mathbf{T}(t) := [T_{(1)}(t) \;\; T_{(2)}(t) \;\; \ldots \;\; T_{(D)}(t)]$ to be the vector gathering all the functions of the expected number of persons residing in departments $1, 2, ..., D$.

### 2.4.1 The case of departments' expansion

By post-multiplying (9) by $\left(\begin{array}{c|c|c|c} \mathbf{1}' & \mathbf{0}' & \ldots & \mathbf{0}' \\ \hline \mathbf{0}' & \mathbf{1}' & & \mathbf{0}' \\ \hline \vdots & & \ddots & \vdots \\ \hline \mathbf{0}' & \mathbf{0}' & \ldots & \mathbf{1}' \end{array}\right)$, where $\mathbf{1} = [1 \;\; 1 \;\; \ldots \;\; 1]$, $\mathbf{0} = [0 \;\; 0 \;\; \ldots \;\; 0]$, we get:

$$
\begin{aligned}
\frac{d\mathbf{T}(t)}{dt} &\overset{(1)}{=} \mathbf{T}(t)\mathbf{R} \; + [M_{(1)}(t) \;\; \ldots \;\; M_{(D)}(t)] \\
&\quad +\mathbf{T}(t)(-diag\{\mathbf{R}\}) - \mathbf{T}(t)(\mathbf{R} - diag\{\mathbf{R}\}) \\
&= [M_{(1)}(t) \;\; \ldots \;\; M_{(D)}(t)] \qquad (11)
\end{aligned}
$$

with $\mathbf{R} = \left(\begin{array}{c|c|c|c} -q_{(1)} & q_{(12)} & \cdots & q_{(1D)} \\ \hline q_{(21)} & -q_{(2)} & & q_{(2D)} \\ \hline \vdots & & \ddots & \vdots \\ \hline q_{(D1)} & q_{(D2)} & \cdots & -q_{(D)} \end{array}\right)$ being the infinitesimal generator matrix of the continuous time Markov chain that governs the evolution of the functions of the departments' sizes and $diag\{\mathbf{R}\}$ being a diagonal matrix having as elements the diagonal elements of $\mathbf{R}$. Thus, Equation (11) provides the differential equations for the functions of the departments' sizes, provided that all departments are expanding (i.e. $M_{(d)}(t) \geq 0$, $d = 1, \ldots, D$).

### 2.4.2 The case of departments' contraction

In what follows we propose a moderate approach in reducing the size of a department, in the sense that no new persons are hired in the particular department, and its employees gradually either abandon the system due to wastage or are allocated to other departments. In this context we have:

**Assumption 2.1.** We accept that the system's administration decides the *contraction* of a department $\delta \in \{1, 2, \ldots, D\}$ if

i. there exists at least one department $b \in \{1, 2, \ldots, D\}$, $b \neq \delta$, such that $q_{(\delta b)} > 0$, that is indivinduals of department $\delta$ have the alternative to move to departments other from the one they reside,

ii. there is a $t_0 \geq 0$ such that $\forall t > t_0$, the management discontinues

   a) hiring to fill empty posts due to persons being transferred to other departments, and

   b) opening new positions in departments $1, 2, \ldots, D$,

   i.e. it holds on the grounds of (6) that $\theta_{(\delta)}(t) := \theta_{0(\delta)}(t) = -\sum_{b=1, b\neq\delta}^{D} T_{(b)}(t) q_{(b\delta)}$.

Notice that under Assumption 2.1 constraint (4) still holds in the form:

$$\mathbf{N}_{(\delta)}(t)\mathbf{w'}_{(\delta)} + T_{(\delta)}(t) \sum_{\substack{m=1 \\ m\neq\delta}}^{D} q_{(\delta m)} \geq \sum_{\substack{b=1 \\ b\neq\delta}}^{D} T_{(b)}(t) q_{(b\delta)} \tag{12}$$

**Lemma 2.1.** *Let the administration of the system decide the contraction of the departments $\eta, \eta + 1, \ldots, D$, $\eta \in \{1, 2, \ldots, D\}$, then the functions of the magnitudes of these departments tend exponentially fast to 0 as $t \to \infty$.*

*Proof.* We first define the vectors $\mathbf{T}_1(t) := [T_{(1)}(t) \ \ldots \ T_{(\eta-1)}(t)]$, $\mathbf{T}_2(t) := [T_{(\eta)}(t) \ \ldots \ T_{(D)}(t)]$ and the matrices $\mathbf{R} := \left(\begin{array}{c} \mathbf{R}_1 \\ \hline \mathbf{R}_2 \end{array}\right)$ and $diag\{\mathbf{R}\} := \left(\begin{array}{c|c} diag\{\mathbf{R}_1\} & \mathbf{0} \\ \hline \mathbf{0} & diag\{\mathbf{R}_2\} \end{array}\right)$. As a consequence, Equation (11) becomes $\forall t > t_0$:

$$\begin{aligned}
\frac{d[\mathbf{T}_1(t) \mid \mathbf{T}_2(t)]}{dt} &= [\mathbf{T}_1(t) \mid \mathbf{T}_2(t)] \left(\begin{array}{c} \mathbf{R}_1 \\ \hline \mathbf{R}_2 \end{array}\right) + [M_{(1)}(t) \ \ \ldots \ \ M_{(\eta-1)}(t) \mid \mathbf{0}] \\
&\quad + [\mathbf{T}_1(t) \mid \mathbf{T}_2(t)] \left(\begin{array}{c|c} -diag\{\mathbf{R}_1\} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \\
&\quad - [\mathbf{T}_1(t) \mid \mathbf{T}_2(t)] \left\{ \left(\begin{array}{c} \mathbf{R}_1 \\ \hline \mathbf{R}_2 \end{array}\right) - \left(\begin{array}{c|c} diag\{\mathbf{R}_1\} & \mathbf{0} \\ \hline \mathbf{0} & diag\{\mathbf{R}_2\} \end{array}\right) \right\} \\
&= [M_{(1)}(t) \ \ \ldots \ \ M_{(\eta-1)}(t) \mid \mathbf{T}_2(t) diag\{\mathbf{R}_2\}]
\end{aligned} \tag{13}$$

From Equation (13), it can be easily seen that the differential equation that governs the function of the magnitude of the contracted department $m$, $m \in \{\eta, \eta+1, \ldots, D\}$ for $t > t_0$ would be:

$$\frac{dT_{(m)}(t)}{dt} + q_{(m)}T_{(m)}(t) = 0 \tag{14}$$

Equation (14) constitutes a first order linear differential equation with solution $T_{(m)}(t) = T_{(m)}(0)e^{-q_{(m)}t}$ which obviously tends exponentially fast to 0. $\square$

The CTMHMS defined in the current section via differential Equations (9), (11) and (13), can be utilized to look into the evolution of a manpower system's departments' structures in the distant future. In this context, the manpower planner will be able to assess in advance how various strategies may affect the system's operation in the long run. This is exactly the issue of investigating the asymptotic behavior of the CTMHMS, that is the examination of the properties of the limiting structure $\mathbf{N}(\infty)$, studied in the following section.

# 3 Asymptotic behavior of the CTMHMS

In the previous section we began with the notion of the CTMHMS and its basic parameters. In the sequel, on the grounds of these parameters, we provided the analysis of the mobility of a system represented by the CTMHMS. This mobility pattern was initially visualized with the aid of Figure 1. Then, it was analytically described via the formulation of constraint (4) as well as the establishment of equations (5), (6), (9) and (10) that govern the expected personnel structure $\mathbf{N}(t)$ of the CTMHMS. Furthermore, subsection 2.4 elaborated on the evolution of the departments' magnitudes $T_{(d)}(t)$, $d = 1, \ldots, D$ under assumptions of departmental expansion or contraction. It was shown that under the departmental contraction assumption constraint (4) takes the form of (12).

The current section builds upon equations (6) to (10) provided that constraint (12) holds, in order to:

(a) examine the conditions under which the expected stock vector of the CTMHMS $\mathbf{N}(t)$ converges, and

(b) specify its limiting structure $\lim_{t\to\infty} \mathbf{N}(t)$ as a function of the basic parameters of the CTMHMS.

In other words, this section concerns the limiting behavior of the CTMHMS. The investigation of this long run (limiting) behavior under specific management strategies helps the system's administration to avoid undesirable future situations by making certain interventions so as to improve and secure the financial viability of the company.

Towards our goal to investigate the limiting expected structure of the CTMHMS, we need to provide conditions for the convergence of the functional components of Equation (10) that give the general form of the structure $\mathbf{N}(t)$ at any time $t \geq 0$. Thus, we need to analyse the limiting behavior of the integral $\int_0^t \boldsymbol{\theta}(x)dx$ when $t \to \infty$ and then continue on the elaboration of the limiting form of $\boldsymbol{\Phi}(t)$ and $\boldsymbol{\Phi}(t-x)$ as $t \to \infty$. In this respect, we provide as a first step a Lemma that gives a sufficient set of conditions for the convergence of the integral $\int_0^\infty \boldsymbol{\theta}(x)dx$. This will enable us to proceed to the main result of this section in Theorem 3.1, that is the sufficient set of conditions for the existence and the determination of the limiting expected structure $\mathbf{N}(\infty)$ of the CTMHMS.

In what follows we will use some preliminary and known results from the literature of stochastic processes. For convenience, this material is gathered in Appendix. In addition, we use the norm of a matrix or a vector as defined in (23) and (24) respectively in Appendix.

**Lemma 3.1.** *A sufficient set of conditions for the convergence of the integral $\int_0^\infty \boldsymbol{\theta}(t)dt$ of the CTMHMS is the following:*

*(a) the rate matrix $\mathbf{R}$ is reducible, having its state space $\{1,\ldots,D\}$ partitioned as $\{1,\ldots,D\} = \{\mathcal{T}_{\mathbf{R}} = \{\eta,\ldots,D\}\} \cup C_{\mathbf{R}1} \cup \ldots \cup C_{\mathbf{R}r}$. $\mathcal{T}_{\mathbf{R}}$ stands for the set formed from all transient departments with $\eta \in \{2,\ldots,D\}$ (i.e. not all the departments are transient), and $C_{\mathbf{R}\nu}$, $\nu = 1,\ldots,r$ denotes the closed irreducible sets comprised of communicating departments. In such a case, $\mathbf{R}$ takes the form*

$$\mathbf{R} = \left(\begin{array}{cccc|c} \mathbf{R}_{C_{\mathbf{R}1}} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{R}_{C_{\mathbf{R}r}} & \mathbf{0} \\ \hline \multicolumn{4}{c|}{\mathbf{R}_{\mathcal{R}}} & \mathbf{R}_{\mathcal{T}} \end{array}\right)$$

*(b) i. the vector $\mathbf{T}_{\mathbf{R}\nu}(t) := \{T_{(\beta)}(t)\}$, $\beta \in C_{\mathbf{R}\nu}$, $\nu = 1,2,\ldots,r$ tends exponentially fast to the specific limit: $\lim_{t\to\infty} \mathbf{T}_{\mathbf{R}\nu}(t) = \mu\boldsymbol{\pi}_{\mathbf{R}\nu} := \mathbf{T}_{\mathbf{R}\nu} := \{T_{(\beta)}\}$, for a value $\mu \in \mathbb{R}$. We accept that $\boldsymbol{\pi}_{\mathbf{R}\nu} = \{\pi_k\}_{k\in C_{\mathbf{R}\nu}}$ constitutes the unique stationary distribution of the irreducible continuous time Markov chain with $\mathbf{R}_{C_{\mathbf{R}\nu}}$ as corresponding rate matrix (see Theorem A.1)*

*ii. $M_{(\beta)}(t)$ is non-negative and monotone non-increasing, $\forall\beta \in C_{\mathbf{R}\nu}$, $\nu = 1,2,\ldots,r$, i.e. the non-transient departments are expanding or being eventually stable*

*(c) the system's administration indents to contract (in the sense of Assumption 2.1) all the transient departments $\eta,\ldots,D$ and constraint (12) holds true.*

*Proof.* We shall show that $\lim_{t\to\infty} \boldsymbol{\theta}(t) = \mathbf{0}$ with exponential rate of convergence which implies that $\left\|\int_0^\infty \boldsymbol{\theta}(t)dt\right\| < \infty$ since if $\|\boldsymbol{\theta}(t)\| \leq ce^{-\beta t}$ for $c$, $\beta \geq 0$ then $\left\|\int_0^\infty \boldsymbol{\theta}(t)dt\right\| \leq \int_0^\infty \|\boldsymbol{\theta}(t)\|\, dt \leq \int_0^\infty ce^{-\beta t}dt < \infty$.

We first concentrate on these elements $\theta_{(m)}(t)$ of the vector $\boldsymbol{\theta}(t)$ which correspond to a *department $m$ that belongs to a closed irreducible subset of* $\mathbf{R}$. Without loss of generality, let $m \in C_{\mathbf{R}1} := \{1,2,\ldots,\alpha\}$.

Then, there will be inflows to $m$ from the other departments of $C_{\mathbf{R}1}$, as well as from the departments of $\mathcal{T}_{\mathbf{R}}$ (on the basis of the reducibility of $\mathbf{R}$). As a consequence, $\theta_{(m)}(t)$, according to (6), equals to :

$$M_{(m)}(t) + T_{(m)}(t) \sum_{b \in C_{\mathbf{R}1} - \{m\}} q_{(mb)} - \sum_{b \in C_{\mathbf{R}1} - \{m\}} T_{(b)}(t) q_{(bm)} - \sum_{b \in \mathcal{T}_{\mathbf{R}}} T_{(b)}(t) q_{(bm)}$$

Hence, in one way to achieve $\lim_{t\to\infty} \theta_{(m)}(t) = 0$ exponentially fast, the next conditions should hold true:

i. $\lim_{t\to\infty} M_{(m)}(t) = 0$ geometrically fast, which holds true due to condition (b) (since $T_m(t)$ converges geometrically fast to a limit, its derivative $M_{(m)}(t)$ will converge geometrically fast to 0),

ii. in the closed irreducible set $C_{\mathbf{R}1}$, the *limiting* rate of inflows in $m$ coming from the other departments of $C_{\mathbf{R}1}$ would be equal to the *limiting* rate of outflows from $m$ towards other departments of $C_{\mathbf{R}1}$, in others words it would hold:

$$\begin{aligned} T_{(m)} \sum_{b \in C_{\mathbf{R}1} - \{m\}} q_{(mb)} &- \sum_{b \in C_{\mathbf{R}1} - \{m\}} T_{(b)} q_{(bm)} \\ &\stackrel{(1)}{=} T_{(m)} q_{(m)} - \sum_{b \in C_{\mathbf{R}1} - \{m\}} T_{(b)} q_{(bm)} = 0 \end{aligned} \tag{15}$$

with exponential rate of convergence. This can be achieved only if $\lim_{t\to\infty} T_{(\kappa)}(t) = T_{(\kappa)} \neq 0$ , $\kappa \in C_{\mathbf{R}1}$, exponentially fast and $T_{(\kappa)}$ are the non-zero solutions of the homogeneous linear system $\mathbf{T_{R1}} \mathbf{R}_{C_{\mathbf{R}1}} = \mathbf{0}$, with $\mathbf{T_{R1}} = [T_{(1)} \;\; T_{(2)} \;\; \dots \;\; T_{(\alpha)}]$. Due to the fact that $\mathbf{R}_{C_{\mathbf{R}1}}$ is a rate matrix, it is well known (Kulkarni 1995) that the above linear system has unique solutions up to a constant of multiplication. Thus, if $\boldsymbol{\pi}_{\mathbf{R}1} = \{\pi_k\}_{k \in C_{\mathbf{R}1}}$ is the unique stationary distribution of the irreducible continuous time Markov chain with $\mathbf{R}_{C_{\mathbf{R}1}}$ as corresponding rate matrix (see also Theorem A.1), then $\mathbf{T_{R1}}$ ought to be equal to $\mathbf{T_{R1}} = \mu \boldsymbol{\pi}_{\mathbf{R}1}$ for a value $\mu \in \mathbb{R}$. In this sense, condition (b) assures that (15) can be satisfied with exponential rate.

iii. the total limiting rate of inflows to $m$ from the transient departments should tend to 0 exponentially fast, i.e. $\lim_{t\to\infty} \left( \sum_{b \in \mathcal{T}_{\mathbf{R}}} T_{(b)}(t) q_{(bm)} \right) = 0$ exponentially fast. This can be realised when $\lim_{t\to\infty} T_{(b)}(t) = 0$, $\forall b \in \mathcal{T}_{\mathbf{R}}$ exponentially fast. In this respect, condition (c) makes sure that $\lim_{t\to\infty} T_{(b)}(t) := T_{(b)} = 0$ with exponential rate of convergence (see Lemma 2.1).

As a consequence, conditions (i-iii) make sure that $\lim_{t\to\infty} \theta_{(\beta)}(t) = 0$ with exponential rate of convergence $\forall \beta \in C_{\mathbf{R}\nu}$, $\nu = 1, 2, ..., r$, that is $\lim_{t\to\infty} \theta_{(\beta)}(t) = 0$ for all the departments that belong to a closed irreducible subset $C_{\mathbf{R}\nu}$.

We now turn our attention to the remaining elements $\theta_{(h)}(t)$ of the vector $\boldsymbol{\theta}(t)$ where $h \in \mathcal{T}_{\mathbf{R}} = \{\eta, \dots, D\}$ (elements of the transient departments). In accordance with condition (c) and Lemma 2.1 there exists a $t_0 > 0$ such that $\forall t > t_0$, $\theta_{(h)}(t) = -\sum_{b=1,b\neq h}^{D} T_{(b)}(t)q_{(bh)} = -\sum_{\beta\in C_{\mathbf{R}\nu}} T_{(\beta)}(t)q_{(\beta h)} - \sum_{b=\eta,b\neq h}^{D} T_{(b)}(t)q_{(bh)}$, $\nu = 1, 2, ..., r$. Due to the fact that $C_{\mathbf{R}\nu}$ are closed sets, it would be true that $q_{(\beta h)} = 0$, $\forall\beta \in C_{\mathbf{R}\nu}$, $\nu = 1, 2, ..., r$ which results to the fact that $-\sum_{\beta\in C_{\mathbf{R}\nu}} T_{(\beta)}(t)q_{(\beta h)} = 0$. Thus, $\theta_{(h)}(t) = -\sum_{b=\eta,b\neq h}^{D} T_{(b)}(t)q_{(bh)}$ and tends to 0 with exponential rate of convergence according to the reasoning developed in (iii) above.

As a result, it holds that

- $\lim_{t\to\infty} \theta_{(\beta)}(t) = 0 \ \forall\beta \in C_{\mathbf{R}\nu}$ and
- $\lim_{t\to\infty} \theta_{(h)}(t) = 0 \ \forall h \in \mathcal{T}_{\mathbf{R}}$

both with exponential rate of convergence. □

Notice at this point the fact that Lemma 3.1 achieves the convergence of $\int_0^\infty \boldsymbol{\theta}(t)dt$ via the exponential convergence of $\boldsymbol{\theta}(t)$ to $\mathbf{0}$, i.e. it holds for the non-transient departments $\beta$, $\beta \in \{1, \dots, \eta - 1\}$:

$$
\begin{aligned}
&\underbrace{M_{(\beta)}(t)}_{\underset{t\to\infty}{\to}\ 0} + T_{(d)}(t) \sum_{\substack{m=1 \\ m\neq\beta}}^{D} q_{(\beta m)} - \sum_{\substack{b=1 \\ b\neq\beta}}^{D} T_{(b)}(t)q_{(b\beta)} \xrightarrow{t\to\infty} 0 \\
&\Leftrightarrow T_{(d)}(t) \sum_{\substack{m=1 \\ m\neq\beta}}^{D} q_{(\beta m)} \xrightarrow{t\to\infty} \sum_{\substack{b=1 \\ b\neq\beta}}^{D} T_{(b)}(t)q_{(b\beta)}
\end{aligned}
$$

The above, reflects the following phenomenon: As the size of the non-transient department $T_{(\beta)}(t)$, becomes eventually stable as the time evolves (condition (b) asserts: $\lim_{t\to\infty} T_{(\beta)}(t) = T_{(\beta)} \Leftrightarrow \lim_{t\to\infty} M_{(\beta)}(t) = 0$), the limiting rate of inflows in department $\beta$ stemming from other departments would become equal to the limiting rate of outflows from $\beta$ towards other departments (*intra-departmental flow rate balance equation*).

We remind the reader that the goal of Section 3 is the determination of the limiting behavior of Equation (10) of Section 2 and the establishment of its structure in a closed functional form. In this respect, we now proceed with the main set of limiting conditions for the convergence of the CTMHMS defined in Section 2, Equation (10), as well as with the determination of its asymptotic structure. The results are provided in the form of the following Theorem 3.1.

**Theorem 3.1.** *Let that for the CTMHMS:*

*i. Conditions (a)-(c) of Lemma 3.1 are met,*

*ii. The rate matrices* $\mathbf{Q}_{(dd)} + \mathbf{w}'_{(d)}\mathbf{p}_{(d)0}$, $d = 1, 2, ..., D$, *are irreducible,* *Then, the expected structure of the CTMHMS of Equation (10) in Section 2 converges as $t \to \infty$ to the limiting structure*

$$\mathbf{N}(\infty) = \lim_{t\to\infty} \mathbf{N}(t) = \left\{ \mathbf{N}(0) + \int_{\tau=0}^{\infty} \boldsymbol{\theta}(\tau) d\tau \right\} (\mathbf{\Phi}^{\infty} \,|\, \mathbf{0}) \tag{16}$$

*with $(\mathbf{\Phi}^{\infty} \,|\, \mathbf{0})$ being the limit of the transition probability matrix $\mathbf{\Phi}(t)$ as $t \to \infty$ that corresponds to the continuous time Markov chain with infinitesimal generator $\bar{\mathbf{Q}}$ of the CTMHMS.*

*Proof.* Due to the fact that the rate matrix $\mathbf{R}$ which regulates the transitions among the departments of the CTMHMS is reducible, as described by condition (a) of Lemma 3.1, and the matrices $\mathbf{Q}_{(dd)} + \mathbf{w}'_{(d)}\mathbf{p}_{(d)0}$, $d = 1, 2, ..., D$, are irreducible, it can be easily seen with the aid of a path diagram (Seneta 1981, 14-20) for the corresponding embedded DTMCs of $\bar{\mathbf{Q}}$ (see Proposition A.1), that the rate matrix $\bar{\mathbf{Q}}$ is also reducible and has the same number of closed irreducible subsets as $\mathbf{R}$. Moreover, these closed subsets of the states of $\bar{\mathbf{Q}}$ are comprised of the corresponding subsets of the departments of $\mathbf{R}$ containing all the states of these departments (since the matrices $\mathbf{Q}_{(dd)} + \mathbf{w}'_{(d)}\mathbf{p}_{(d)0}$, $d = 1, 2, ..., D$, are irreducible). Hence, if the state space, say, $\{S_{(1)}, \ldots, S_{(D)}\}$ of $\bar{\mathbf{Q}}$ is partitioned as $\{S_{(1)}, \ldots, S_{(D)}\} = \mathcal{T}_{\bar{\mathbf{Q}}} \cup C_{\bar{\mathbf{Q}}1} \cup ... \cup C_{\bar{\mathbf{Q}}r}$ with $\mathcal{T}_{\bar{\mathbf{Q}}} = \{S_{(\eta)}, \ldots, S_{(D)}\}$ (by reason of $\mathcal{T}_{\mathbf{R}} = \{\eta, \ldots, D\}$) standing for the set of all the states of the system's transient departments and $C_{\bar{\mathbf{Q}}\nu}$, $\nu = 1, ..., r$ denoting the closed irreducible sets including all the states of the specific departments that form irreducible sets. In this context, $\bar{\mathbf{Q}}$ can be written with suitable reallocation of its elements in the corresponding partitioned form, and the relevant vector of the system's expected structure $\mathbf{N}(t)$ would take the form $\mathbf{N}(t) = [\mathbf{N}_{C_{\bar{\mathbf{Q}}1}}(t) \mid \ldots \mid \mathbf{N}_{C_{\bar{\mathbf{Q}}r}}(t) \mid \underbrace{\mathbf{N}_{(\eta)}(t) \mid \ldots \mid \mathbf{N}_{(D)}(t)}_{:=\mathbf{N}_{\mathcal{T}_{\bar{\mathbf{Q}}}}(t)}]$. In a similar manner, elements of $\boldsymbol{\theta}(t)$ of (9) can be reallocated accordingly so that Equation (10) holds true.

Since $\bar{\mathbf{Q}}$ is a reducible rate matrix, in view of Theorem A.1 it will hold:

$$\lim_{t\to\infty} \mathbf{\Phi}(t) = \left( \begin{array}{cccc|c} \mathbf{\Pi}_{C_{\bar{\mathbf{Q}}1}} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{\Pi}_{C_{\bar{\mathbf{Q}}r}} & \mathbf{0} \\ \hline & & \mathbf{A} & & \mathbf{0} \end{array} \right) := (\bar{\mathbf{\Phi}}^{\infty} \,|\, \mathbf{0}) \tag{17}$$

where $\mathbf{\Pi}_{C_{\bar{\mathbf{Q}}\nu}} = \begin{pmatrix} \boldsymbol{\pi}_{C_{\bar{\mathbf{Q}}\nu}} \\ \vdots \\ \boldsymbol{\pi}_{C_{\bar{\mathbf{Q}}\nu}} \end{pmatrix}$, with $\boldsymbol{\pi}_{C_{\bar{\mathbf{Q}}\nu}} = \{\pi_k\}_{k \in C_{\bar{\mathbf{Q}}\nu}}$ the unique stationary distribution of the irreducible CTMC with $C_{\bar{\mathbf{Q}}\nu}$ as state space and $\mathbf{A}$ the matrix consisting of the corresponding limiting probabilities from the transient states to the states of the closed sets (absorbing states) as given by (27).

If we return now to Equation (10):

$$\mathbf{N}(t) = \mathbf{N}(0)\mathbf{\Phi}(t) + \int_0^t \boldsymbol{\theta}(\tau)\mathbf{\Phi}(t-\tau) d\tau$$

notice that due to the fact that $\theta_{(d)}(t) := M_{(d)}(t) + T_{(d)}(t)\sum_{m=1,m\neq d}^{D} q_{(dm)} - \sum_{b=1,b\neq d}^{D} T_{(b)}(t)q_{(bd)}$, $d = 1,\ldots,D$ constraint (4) does not ensure that $\theta_{(d)}(t) \geq 0 \;\; \forall t \geq 0$. Hence, $\theta_{(d)}(t)$ might be negative for some $t \geq 0$. Therefore, let:

$$\theta^{+}_{(d)}(t) := \begin{cases} \theta_{(d)}(t), & \text{if } \theta_{(d)}(t) > 0 \\ 0, & \text{if } \theta_{(d)}(t) \leq \; 0 \end{cases}$$

$$\theta^{-}_{(d)}(t) := \begin{cases} -\theta_{(d)}(t), & \text{if } \theta_{(d)}(t) < 0 \\ 0, & \text{if } \theta_{(d)}(t) \geq \; 0 \end{cases}$$

and $\boldsymbol{\theta}^{+}(t) = \{\theta^{+}_{(d)}(t)\}$ as well as $\boldsymbol{\theta}^{-}(t) = \{\theta^{-}_{(d)}(t)\}$.

With the aid of the above, we get for the second term of (10) that $\forall t \geq 0$ :

$$\int_0^t \boldsymbol{\theta}(\tau)\boldsymbol{\Phi}(t-\tau)d\tau \;\; = \;\; \int_{\tau=0}^{t} \boldsymbol{\theta}^{+}(\tau)\boldsymbol{\Phi}(t-\tau)d\tau - \int_{\tau=0}^{t} \boldsymbol{\theta}^{-}(\tau)\boldsymbol{\Phi}(t-\tau)d\tau \tag{18}$$

At this point notice that $\boldsymbol{\Phi}(t)$ is stochastic. Therefore, $\|\boldsymbol{\Phi}(t-\tau)\| \leq 1 \; \forall t,\tau \geq 0$, such that $t-\tau \geq 0$. In addition, from (17) we have that $\lim_{t\to\infty} \boldsymbol{\Phi}(t-\tau) = \left(\bar{\boldsymbol{\Phi}}^{\infty} \,\middle|\, \mathbf{0}\right)$, $\forall \tau \geq 0$ , such that $t-\tau \geq 0$. Furthermore, $\boldsymbol{\theta}^{+}(t), \boldsymbol{\theta}^{-}(t) \geq \mathbf{0}$, and from Lemma 3.1 we have that $\int_0^\infty \boldsymbol{\theta}(\tau)d\tau$ converges and thus both $\int_0^\infty \boldsymbol{\theta}(\tau)^{+}d\tau$ and $\int_0^\infty \boldsymbol{\theta}(\tau)^{-}d\tau$ converge. As a consequence, the conditions of Theorem A.2 are satisfied for the convergence of the two terms of (18):

$$\begin{aligned} &\lim_{t\to\infty} \textstyle\int_0^t \boldsymbol{\theta}(\tau)\boldsymbol{\Phi}(t-\tau)d\tau \\ &\quad = \lim_{t\to\infty} \int_{\tau=0}^{t} \boldsymbol{\theta}^{+}(\tau)\boldsymbol{\Phi}(t-\tau)d\tau - \lim_{t\to\infty} \int_{\tau=0}^{t} \boldsymbol{\theta}^{-}(\tau)\boldsymbol{\Phi}(t-\tau)d\tau \\ &\quad \overset{\text{Theorem A.2}}{=} \int_0^\infty \boldsymbol{\theta}^{+}(\tau)\left(\bar{\boldsymbol{\Phi}}^{\infty} \,\middle|\, \mathbf{0}\right)d\tau - \int_0^\infty \boldsymbol{\theta}^{-}(\tau)\left(\bar{\boldsymbol{\Phi}}^{\infty} \,\middle|\, \mathbf{0}\right)d\tau \\ &\quad = \int_0^\infty \boldsymbol{\theta}(\tau)\left(\bar{\boldsymbol{\Phi}}^{\infty} \,\middle|\, \mathbf{0}\right)d\tau \text{ with } \left\| \int_0^\infty \boldsymbol{\theta}(\tau)\left(\bar{\boldsymbol{\Phi}}^{\infty} \,\middle|\, \mathbf{0}\right)d\tau \right\| < \infty \end{aligned} \tag{19}$$

Then, we get for the limiting structure of the CTMHMS, through (10) and (19):

$$\begin{aligned} \lim_{t\to\infty} \mathbf{N}(t) &= \lim_{t\to\infty} \mathbf{N}(0)\boldsymbol{\Phi}(t) + \lim_{t\to\infty} \int_0^t \boldsymbol{\theta}(\tau)\boldsymbol{\Phi}(t-\tau)d\tau \\ &\overset{(19)}{=} \mathbf{N}(0)\left(\bar{\boldsymbol{\Phi}}^{\infty} \,\middle|\, \mathbf{0}\right) + \int_0^\infty \boldsymbol{\theta}(\tau)\left(\bar{\boldsymbol{\Phi}}^{\infty} \,\middle|\, \mathbf{0}\right)d\tau \\ &= \left\{\mathbf{N}(0) + \int_{\tau=0}^{\infty} \boldsymbol{\theta}(\tau)d\tau\right\}\left(\boldsymbol{\Phi}^{\infty} \,|\, \mathbf{0}\right) \end{aligned} \tag{20}$$

□

We now proceed by providing an arithmetic example that illustrates the aforementioned results.

# 4 Numerical Illustration

In the present section we demonstrate a numerical application that confirms the theoretical results of the previous sections. The example that follows relies on a manpower system of a hypothetical organization with $D = 3$ departments. In this sense, let the first department ($d = 1$) be comprised of $k_{(1)} = 3$ grades, the second one of $k_{(2)} = 2$ grades and the third one by $k_{(3)} = 3$ grades. Moreover, assume that the system's initial state is $\mathbf{N}(0) = [N_{(1),1}(0) \quad N_{(1),2}(0) \quad N_{(1),3}(0) \mid N_{(2),1}(0) \quad N_{(2),2}(0) \mid N_{(3),1}(0) \quad N_{(3),2}(0) \quad N_{(3),3}(0)] = [60 \quad 35 \quad 10 \mid 35 \quad 10 \mid 20 \quad 10 \quad 4]$ and as a consequence $T_{(1)}(0) = 105$, $T_{(2)}(0) = 45$ and $T_{(3)}(0) = 34$.

As far as the inter-mobility is concerned, we presume that there are transitions from department 1 to 2 with the relevant rate being $q_{(12)} = 1$ and the corresponding distribution of the employees governed by the probability matrix $\mathbf{P}_{(12)} = \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \\ 0 & 1 \end{pmatrix}$. Analogously, suppose that $q_{(13)} = 0$ (there are no movements from department 1 to 3), $q_{(21)} = 2$ with $\mathbf{P}_{(21)} = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0 & 0.8 & 0.2 \end{pmatrix}$, $q_{(23)} = 0$, $q_{(31)} = 4$ with $\mathbf{P}_{(31)} = \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \\ 0 & 0 & 1 \end{pmatrix}$, and $q_{(32)} = 3$ with $\mathbf{P}_{(32)} = \begin{pmatrix} 1 & 0 \\ 0.9 & 0.1 \\ 0 & 1 \end{pmatrix}$. As a result, the infinitesimal generator matrix that rules the evolution of the departments' sizes is:

$$\mathbf{R} = \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \\ \hline 4 & 3 & -7 \end{array}\right) = \left(\begin{array}{c|c} \mathbf{R}_{C_{\mathbf{R}1}} & \mathbf{0} \\ \hline \mathbf{R}_{\mathcal{R}} & \mathbf{R}_{\mathcal{T}} \end{array}\right) \tag{21}$$

hence, the rate matrix $\mathbf{R}$ is reducible and its state space can be partitioned: $\{1, 2, 3\} = \{\mathcal{T}_{\mathbf{R}} = \{3\} \cup C_{\mathbf{R}1} = \{1, 2\}\}$, where $C_{\mathbf{R}1}$ is unique closed irreducible set of departments with corresponding rate matrix $\mathbf{R}_{C_{\mathbf{R}1}}$.

Think now the wastage and the intra-mobility of the system. Assume that the vectors gathering the leaving intensities of the system's departments are equal to $\mathbf{w}_{(1)} = [6 \quad 7 \quad 9]$, $\mathbf{w}_{(2)} = [4 \quad 6]$, and $\mathbf{w}_{(3)} = [2 \quad 4 \quad 8]$ and the matrices of the intra-mobility intensities are:

$$\mathbf{Q}_{(11)} = \begin{pmatrix} -17 & 8 & 2 \\ 1 & -13 & 4 \\ 0 & 2 & -12 \end{pmatrix} \quad \mathbf{Q}_{(22)} = \begin{pmatrix} -18 & 12 \\ 2 & -10 \end{pmatrix} \quad \mathbf{Q}_{(33)} = \begin{pmatrix} -34 & 15 & 10 \\ 3 & -22 & 8 \\ 0 & 3 & -18 \end{pmatrix}$$

Furthermore, we assume that the the input probabilities vectors are: $\mathbf{p}_{(1)0} = [0.7 \quad 0.2 \quad 0.1]$, $\mathbf{p}_{(2)0} =$

$[0.85\ \ 0.15]$, and $\mathbf{p}_{(3)0} = [1\ \ 0\ \ 0]$. On the grounds of the above, the rate matrix $\bar{\mathbf{Q}}$ equals:

$$\bar{\mathbf{Q}} = \left(\begin{array}{ccc|cc|ccc} -12.8 & 9.2 & 2.6 & 0.8 & 0.2 & 0 & 0 & 0 \\ 5.9 & -11.6 & 4.7 & 0.5 & 0.5 & 0 & 0 & 0 \\ 6.3 & 3.8 & -11.1 & 0 & 1 & 0 & 0 & 0 \\ \hline 1.4 & 0.6 & 0 & -14.6 & 12.6 & 0 & 0 & 0 \\ 0 & 1.6 & 0.4 & 7.1 & -9.1 & 0 & 0 & 0 \\ \hline 3.6 & 0.4 & 0 & 3 & 0 & -32 & 15 & 10 \\ 0 & 3.6 & 0.4 & 2.7 & 0.3 & 7 & -22 & 8 \\ 0 & 0 & 4 & 0 & 3 & 8 & 3 & -18 \end{array}\right)$$

Note that due to the (reducible) form the rate matrix $\mathbf{R}$ and the irreducibility of the matrices $\mathbf{Q}_{(dd)}$, $d = 1, 2, 3$, the matrix $\bar{\mathbf{Q}}$ is reducible with one closed irreducible set comprised of the states of departments 1 and 2, and one closed set of the transient states of department 3. Then, according to Theorem A.1, it will be true that

$$\lim_{t\to\infty} \mathbf{\Phi}(t) = (\mathbf{\Phi}^{\infty} \,|\, \mathbf{0}) = \left(\begin{array}{ccccc|ccc} 0.223306 & 0.269416 & 0.173945 & 0.123503 & 0.20983 & 0 & 0 & 0 \\ 0.223306 & 0.269416 & 0.173945 & 0.123503 & 0.20983 & 0 & 0 & 0 \\ 0.223306 & 0.269416 & 0.173945 & 0.123503 & 0.20983 & 0 & 0 & 0 \\ 0.223306 & 0.269416 & 0.173945 & 0.123503 & 0.20983 & 0 & 0 & 0 \\ 0.223306 & 0.269416 & 0.173945 & 0.123503 & 0.20983 & 0 & 0 & 0 \\ 0.223306 & 0.269416 & 0.173945 & 0.123503 & 0.20983 & 0 & 0 & 0 \\ 0.223306 & 0.269416 & 0.173945 & 0.123503 & 0.20983 & 0 & 0 & 0 \\ 0.223306 & 0.269416 & 0.173945 & 0.123503 & 0.20983 & 0 & 0 & 0 \end{array}\right)$$

Moreover, let the administration of the system decide *the contraction of the third department* and its eventual amalgamation with the other two departments, in the context of Assumption 2.1. In addition, the administration also suggests the personnel sizes of the first and the second department become eventually 130 and 65 respectively. *Notice* that these (limiting) target sizes meet condition (b) of Lemma 3.1, due to the fact $[130\ \ 65] = \lim_{t\to\infty} \mathbf{T}_{\mathbf{R}1}(t) = \mu \boldsymbol{\pi}_{\mathbf{R}1} = 195 \cdot [2/3\ \ 1/3]$ and $[2/3\ \ 1/3]$ is the stationary distribution of the unique closed irreducible set of departments with corresponding rate matrix $\mathbf{R}_{C_{\mathbf{R}1}}$. Towards the achievement of the above departmental size goals, the administration sets the rates of increase for the three departments to be:

1. $M_{(1)}(t) = 12.5e^{-0.5t}$ (i.e. the total increase of department 1 in the interval $[0, +\infty)$ would be $\int_0^{\infty} 12.5e^{-0.5t}dt = 25$ so that size of department 1 goes from 105 to 130),

2. $M_{(2)}(t) = 10e^{-0.5t}$ (total increase of department 2 is 20),

3. $M_{(3)}(t) = 0$ by definition, since department 3 would be contracted (see Assumption 2.1),

and thus, for $t \geq 0$, the magnitude of the first department would be equal to $T_1(t) = 105 + \int_0^t 12.5e^{-0.5x}dx$, of the second $T_2(t) = 45 + \int_0^t 10e^{-0.5x}dx$, and of the third $T_3(t) = 34e^{-7t}$ (see the proof of Lemma 2.1). Then, according to Equation (6) it would be:

$$
\begin{aligned}
\theta_{(1)}(t) &= 12.5e^{-0.5t} + 1 \cdot \left(105 + \int_0^t 12.5e^{-0.5x}dx\right) - 2 \cdot \left(45 + \int_0^t 10e^{-0.5x}dx\right) \\
&\quad -4 \cdot \left(34e^{-7t}\right) \\
\theta_{(2)}(t) &= 10e^{-0.5t} + 2 \cdot \left(45 + \int_0^t 10e^{-0.5x}dx\right) - 1 \cdot \left(105 + \int_0^t 12.5e^{-0.5x}dx\right) \\
&\quad -3 \cdot \left(34e^{-7t}\right) \\
\theta_{(3)}(t) &= 0
\end{aligned}
$$

and on the basis of the above, the values of the vector $\boldsymbol{\theta}(t)$ are calculated accordingly. Then we compute $\int_0^\infty \boldsymbol{\theta}(t)dt = [24.9 \;\; 7.11429 \;\; 3.55714 \;\; | \;\; -20.8857 \;\; -3.68571 \;\; | \;\; 0 \;\; 0 \;\; 0]$.

As it can be easily seen, the conditions of Theorem 3.1 are met and therefore the theoretical limiting structure, in accordance with (16), equals to

$$\mathbf{N}(\infty) = [43.5446 \;\; 52.5362 \;\; 33.9192 \;\; | \;\; 24.0831 \;\; 40.9169 \;\; | \;\; 0 \;\; 0 \;\; 0] \tag{22}$$

Apparently, $T_{(1)}(\infty) = 130$, $T_{(2)}(\infty) = 65$ and $T_{(3)}(\infty) = 0$.

Next, by means of Wolfram Mathematica (version 11.1) (2017) we calculated numerically the probability matrix of interval probabilities $\mathbf{\Phi}(t) = e^{\bar{\mathbf{Q}}t} := \sum_{\kappa=0}^{\infty} \frac{(\bar{\mathbf{Q}}t)^\kappa}{\kappa!}$. With the aid of Equation (10), we obtained Table 1 depicting at certain time points $t \in [0, \infty)$ the expected structure of the system $\mathbf{N}(t)$. Notice that the timepoints $t \geq 0$ at which the structure of the system $\mathbf{N}(t)$ were calculated on Table 1, were chosen arbitrarily, without loss of generality, in order to show some random snapshots of the path of the structures $\mathbf{N}(t)$ until their convergence to $\mathbf{N}(\infty)$. The structure $\mathbf{N}(t)$ could have been calculated for any $t \geq 0$.

We are also concerned with the efficiency of the convergence to $\mathbf{N}(\infty)$. In the literature, there are several approaches cornering the measurement of efficiency in a manpower planning model. Most of them are employed in attainability (control, in general) problems in an attempt to measure the efficiency of a recruitment strategy-policy under some specific criterion. For example, in the work of De Feyter, Guerry, and Komarudin (2017), the authors introduced two measures to evaluate the cost-effectiveness of a recruitment strategy expressed as a combination of the cost ratio and the desirability degree. In addition, Udom (2014) proposed a 2-norm optimality criterion which is equivalent to a minimum effort criterion to obtain a 2-norm optimal control for the system. From another viewpoint, Yadavalli, Natarajan, and Udayabhaskaran (2002) defined an efficiency criterion for the personnel which when exceeds a threshold, leads to the promotion of an employee. In other words, the efficiency criterion depends on the problem at hand. Thus, in order to measure the efficiency that concerns the convergence of $\mathbf{N}(t)$, we resort to measuring at any timepoint $t$, for which the structure of the system $\mathbf{N}(t)$ was calculated on Table 1, the following:

a. how close is the structure $\mathbf{N}(t)$ to its limiting structure $\mathbf{N}(\infty)$,

b. how much the current structure has changed in comparison to the previous one observed on Table 1.

The above measurements were made with the aid of the Euclidean Distance as a metric(norm). In this respect, on Table 1, norm1($t$)= $\|\mathbf{N}(t) - \mathbf{N}(\infty)\|$ for any $t \geq 0$, whereas, for two consecutively observed instances, norm2(0.01)=$\|\mathbf{N}(0.01) - \mathbf{N}(0)\|$, norm2(0.03)=$\|\mathbf{N}(0.03) - \mathbf{N}(0.01)\|$ and so on. As one can see from Table 1, the structure of the system, after time point $t = 27$, coincides with the theoretical limiting structure of (22). Moreover, the two norms constitute two measures of the efficiency of the system's convergence. Finally, we observed that at every time point of the illustration it held that $[r_{(1)}(t) \mid r_{(2)}(t) \mid r_{(3)}(t)] \geq 0$ and thus constraints (4) were not violated.

Table 1: Numerical illustration of the evolution of the expected structure of a CTMHMS.

| Time $t$ | Structures $\mathbf{N}(t)$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $[N_{(1),1}(t)$ | $N_{(1),2}(t)$ | $N_{(1),3}(t)$ \| | $N_{(2),1}(t)$ | $N_{(2),2}(t)$ \| | $N_{(3),1}(t)$ | $N_{(3),2}(t)$ | $N_{(3),3}(t)]$ | norm1($t$) | norm2($t$) |
| 0.01 | [56.1512 | 36.9713 | 12.0022 \| | 32.0106 | 13.0891 \| | 15.4813 | 10.5178 | 5.7023] | 45.8431000 | 8.04805000 |
| 0.03 | [50.0145 | 39.7711 | 15.5867 \| | 27.4444 | 17.8534 \| | 10.0414 | 10.0785 | 7.4399] | 36.631600 | 11.6061000 |
| 0.05 | [45.5997 | 41.4562 | 18.5614 \| | 24.2981 | 21.1957 \| | 7.1883 | 8.8981 | 7.8729] | 30.73530000 | 7.872090000 |
| 0.07 | [42.4883 | 42.4173 | 20.9542 \| | 22.1460 | 23.5419 \| | 5.5571 | 7.6245 | 7.6476] | 26.92100000 | 5.549810000 |
| 0.1 | [39.5217 | 43.0816 | 23.6160 \| | 20.1397 | 25.8357 \| | 4.1306 | 5.9635 | 6.7897] | 23.52550000 | 5.580590000 |
| 0.3 | [36.3670 | 43.5726 | 28.5426 \| | 17.9451 | 29.8407 \| | 0.9685 | 1.3439 | 1.8511] | 18.09140000 | 10.53810000 |
| 0.5 | [37.0534 | 44.4316 | 29.0449 \| | 18.4021 | 31.0219 \| | 0.2389 | 0.3304 | 0.4574] | 16.19070000 | 2.562630000 |
| 0.7 | [37.6930 | 45.2404 | 29.4494 \| | 18.9227 | 31.9835 \| | 0.0589 | 0.0814 | 0.1128] | 14.62560000 | 1.623470000 |
| 1 | [38.5146 | 46.2697 | 30.0524 \| | 19.6364 | 33.2330 \| | 0.0007 | 0.0099 | 0.0138] | 12.58320000 | 2.046040000 |
| 3 | [41.6946 | 50.2317 | 32.4955 \| | 22.4469 | 38.0905 \| | 0.0000 | 0.0000 | 0.0000] | 4.628840000 | 7.954410000 |
| 5 | [42.8640 | 51.6884 | 33.3954 \| | 23.4812 | 39.8771 \| | 0.0000 | 0.0000 | 0.0000] | 1.7028600000 | 2.925980000 |
| 7 | [43.2942 | 52.2243 | 33.7265 \| | 23.8617 | 40.5344 \| | 0.0000 | 0.0000 | 0.0000] | 0.6264470000 | 1.076410000 |
| 9 | [43.4525 | 52.4215 | 33.8483 \| | 24.0017 | 40.7762 \| | 0.0000 | 0.0000 | 0.0000] | 0.2304590000 | 0.395988000 |
| 11 | [43.5107 | 52.4940 | 33.8931 \| | 24.0532 | 40.8651 \| | 0.0000 | 0.0000 | 0.0000] | 0.0847830000 | 0.145676000 |
| 15 | [43.5400 | 52.5305 | 33.9157 \| | 24.0791 | 40.9099 \| | 0.0000 | 0.0000 | 0.0000] | 0.0114768000 | 0.073306300 |
| 19 | [43.5440 | 52.5354 | 33.9187 \| | 24.0826 | 40.9159 \| | 0.0000 | 0.0000 | 0.0000] | 0.0015563500 | 0.009920930 |
| 21 | [43.5444 | 52.5359 | 33.9190 \| | 24.0829 | 40.9165 \| | 0.0000 | 0.0000 | 0.0000] | 0.0005757170 | 0.000981557 |
| 24 | [43.5446 | 52.5362 | 33.9191 \| | 24.0831 | 40.9168 \| | 0.0000 | 0.0000 | 0.0000] | 0.0001367760 | 0.000443782 |
| 26 | [43.5446 | 52.5362 | 33.9192 \| | 24.0831 | 40.9168 \| | 0.0000 | 0.0000 | 0.0000] | 0.0000646158 | 0.000080571 |
| $t \geq 27$ | [43.5446 | 52.5362 | 33.9192 \| | 24.0831 | 40.9169 \| | 0.0000 | 0.0000 | 0.0000] | 0.0000517142 | 0.00001845 |

# 5 Concluding remarks

In this work a Markov manpower planning model, namely the Continuous Time Multi-level Homogeneous Manpower System (CTMHMS) was introduced and analyzed. After establishing its functional form, its asymptotic behavior was studied under a continuous time scale. In this context, we introduced the parameters and the corresponding relations that describe the evolution of its expected personnel structures, and then looked into the conditions that drive the system to limiting personnel stocks by assimilating the concept of department contraction, a means of mild control exerted on the system.

The CTMHMS encompasses the potential movements of the employees of an organization both at vertical and horizontal level as in Ossai and Uche (2009), Guerry and De Feyter (2012) and Dimitriou, Georgiou, and Tsantas (2013). In this sense, the suggested system can be used to project the personnel's mobility not only inside various departments but also among these departments, a mobility which is meant to achieve economies of scale as well as, often, economies of scope. More generally, a framework like the CTMHMS can be utilized by the Human Resource Management to simulate various scenarios and optimization procedures regarding the policies of external recruitment, the impacts of potential transfers of employees among departments and the feasible points in time that each policy or departmental transfer should be applied. Next, this information along with the acknowledgement of the limiting personnel stocks, which reveals future potential weaknesses of the system, should be assessed by the management so as to reach satisfying decisions concerning the efficient functional and financial operation of the system.

Areas of further research for this work might include the investigation of non-homogeneous models, the use of truncated distributions within intra or inter mobility transfers, or even an attempt to incorporate a risk-based approach as suggested by Jaillet, Loke, and Sim (2018). Other ideas that may be more readily developed concern the possibility of looking into the sizes of the departments along with the sizes of personnel stocks as optimization problems (maintainability, attainability) or even balancing attainability with promotion steadiness as in Komarudin et al. (2015).

# Appendix

In the current appendix we present preliminary concepts and results that are required for the study of the limiting behavior of the CTMHMS in Section 3. In this respect, we define the norm of a matrix $\mathbf{A} = \{a_{ij}\}$ as

$$\|\mathbf{A}\| = \sup_i \sum_j |a_{ij}| \tag{23}$$

and the norm of a vector $\mathbf{u} = \{u_i\}$ as

$$\|\mathbf{u}\| = \sup_i |u_i| \tag{24}$$

The following results can be found in various textbooks on stochastic processes such as Kulkarni

(1995).

**Proposition A.1.** *Let $\{X(t), t \geq 0\}$ be a continuous time Markov chain (CTMC) with rate matrix $\mathbf{Q} = \{q_{ij}\}$ and $\{X_n, n \in \mathbb{N}\}$ be its embedded discrete time Markov chain (DTMC) with transition probability matrix $\mathbf{P} = \{p_{ij}\}$ defined as*

$$p_{ij} = \begin{cases} \frac{q_{ij}}{\sum_{j\neq i} q_{ij}} & \text{if } \sum_{j\neq i} q_{ij} \neq 0,\ i \neq j, \\ 0 & \text{if } \sum_{j\neq i} q_{ij} \neq 0,\ i = j, \\ 0 & \text{if } \sum_{j\neq i} q_{ij} = 0,\ i \neq j, \\ 1 & \text{if } \sum_{j\neq i} q_{ij} = 0,\ i = j. \end{cases}$$

*Then the following statements hold*

*a. a state $i$ communicates with $j$ ($i \leftrightarrow j$) for $\{X(t), t \geq 0\}$ if and only if $i \leftrightarrow j$ for $\{X_n, n \in \mathbb{N}\}$,*

*b. $C$ is a (closed) communicating class for $\{X(t), t \geq 0\}$ if and only if is a (closed) communicating class for $\{X_n, n \in \mathbb{N}\}$,*

*c. $\{X(t), t \geq 0\}$ is irreducible if and only if $\{X_n, n \in \mathbb{N}\}$ is irreducible,*

*d. a state $i$ is recurrent (transient) for the CTMC if and only if it is recurrent (transient) for the embedded DTMC.*

**Proposition A.2.** *Let an irreducible CTMC with state space $S$ and the corresponding DTMC as defined in Proposition A.1. Let $\boldsymbol{\pi}$ be the positive solution to $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$. Then the CTMC is positive recurrent if and only if*

$$\sum_{i\in S} \frac{\pi_i}{\sum_{j\neq i} q_{ij}} < \infty$$

**Corollary A.1.** *According to Proposition A.2 an irreducible CTMC with finite state space is always positive recurrent.*

Let a reducible CTMC, with finite state space $S$, governed by the rate matrix $\mathbf{Q} = \{q_{ij}\}_{i,j\in S}$, having the embedded DTMC as described by Proposition A.1, and transition probability matrix $\boldsymbol{\Phi}(t) = \{\phi_{ij}(t)\}$, $t \geq 0$. In such a case, $\mathbf{Q}$ can be rewritten in a partitioned form with its state space being $S = \mathcal{T} \cup C_1 \cup ... \cup C_r$, where $\mathcal{T}$ denotes the set of all transient classes and $C_\nu$, $\nu = 1, ..., r$ the $\nu$-th closed irreducible set consisting solely of positive recurrent states (Corollary A.1). Then, the partitioned form of $\mathbf{Q}$ would be comprised of the matrices $\mathbf{Q}_{C_v}$, $\nu = 1, ..., r$ corresponding to the rate matrices (their rows sum to zero) of the closed irreducible subsets $C_\nu$ of $S$, of $\mathbf{Q}_{\mathcal{R}}$ being a non negative matrix, and of $\mathbf{Q}_{\mathcal{T}}$

corresponding to the rate matrix (with at least one row sum less than 0) of transient states. Thus,

$$\mathbf{Q} = \left( \begin{array}{cccc|c} \mathbf{Q}_{C_1} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{Q}_{C_r} & \mathbf{0} \\ \hline \multicolumn{4}{c|}{\mathbf{Q}_{\mathcal{R}}} & \mathbf{Q}_{\mathcal{T}} \end{array} \right) \tag{25}$$

with corresponding embedded DTMC having the form

$$\mathbf{P} = \left( \begin{array}{cccc|c} \mathbf{P}_{C_1} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{P}_{C_r} & \mathbf{0} \\ \hline \multicolumn{4}{c|}{\mathbf{P}_{\mathcal{R}}} & \mathbf{P}_{\mathcal{T}} \end{array} \right) \tag{26}$$

Then, the next theorem holds true:

**Theorem A.1.** *For the above reducible CTMC it holds*

$$\lim_{t\to\infty} \phi_{ij}(t) = \Pi_{ij},\ i,j \in S$$

*or equivalently in matrix form*

$$\lim_{t\to\infty} \mathbf{\Phi}(t) = \mathbf{\Pi}$$

*with,* $\Pi_{ij} = 0$ *if* $j \in \mathcal{T}$, *and for any* $j \in C_\nu$, $\nu = 1, ..., r$

$$\Pi_{ij} = \begin{cases} \pi_j, & i \in C_\nu,\ \nu = 1, ..., r; \\ 0, & i \in C_{\nu'},\ \nu' \neq \nu,\ \nu' = 1, ..., r; \\ a_{i,C_\nu}\pi_j, & i \in \mathcal{T}. \end{cases} \tag{27}$$

*where* $\boldsymbol{\pi} = \{\pi_k\}_{k\in C_\nu}$ *is the unique stationary distribution of the irreducible CTMC with state space* $C_\nu$, *determined by the unique solution to* $\boldsymbol{\pi}\mathbf{Q}_{C_\nu} = \mathbf{0}$ *and* $\boldsymbol{\pi}\mathbf{1}' = 1$, *and* $a_{i,C_\nu}$, $i \in \mathcal{T}$ *is the absorption probability from state* $i$ *to the set* $C_\nu$ *determined by the unique solution of the linear system* $y_i - \sum_{j\in\mathcal{T},j\neq i} p_{ij}y_j = \sum_{j\in C_\nu} p_{ij}$, $i \in \mathcal{T}$. *Equivalently,* $a_{i,C_\nu}$ *can be obtained (Isaacson and Madsen 1976, Kemeny and Snell 1976) by adding the corresponding elements of the matrix* $(\mathbf{I} - \mathbf{P}_{\mathcal{T}})^{-1}\mathbf{P}_{\mathcal{R}}$ *consisting of the absorption probabilities for the embedded DTMC.*

Finally, it is important for the understanding Section 3, to recall the following theorem:

**Theorem A.2.** (Matrix form of Bounded Convergence Theorem (Rudin 1964)) *Let* $\mathbf{B}(t)$, $t \geq 0$ *be a function of non-negative, finite dimensional matrices and assume that for the function of finite dimen-*

*sional matrices* $\mathbf{A}(t,s)$, $t,s \geq 0$ *the limit*

$$\lim_{s\to\infty} \mathbf{A}(t,s) = \mathbf{A}(t) < \infty$$

*exists for every* $t \geq 0$. *Then if there exists a finite number* $M > 0$ *such that* $\|\mathbf{A}(t,s)\| \leq M\ \forall t,s \geq 0$ *and* $\mathbf{B} = \int_0^\infty \mathbf{B}(x)dx$, *with* $\|\mathbf{B}\| < \infty$, *it would be true that*

$$\lim_{t\to\infty} \int_0^t \mathbf{B}(x)\mathbf{A}(x,t)dx = \int_0^\infty \mathbf{B}(x)\left(\lim_{t\to\infty} \mathbf{A}(x,t)\right)dx = \int_0^\infty \mathbf{B}(x)\mathbf{A}(x)dx.$$

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