Preprint 2019

QUANTIFYING INTERACTIONS IN NONLINEAR FEEDBACK DYNAMICS: A TIME-SERIES ANALYSIS

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In this paper, we further study the dynamics of the Kyrtsou model composed of heterogeneous nonlinear feedback rules. For various levels and types of underlying nonlinearity, we analyse the resulting time series by means of the largest Lyapunov exponent. Our results highlight that the observed interaction among feedback mechanisms cannot lead to a univocal interpretation of system complexity.

Keywords: Nonlinear time series analysis, Interaction, Feedback dynamics, Largest Lyapunov Exponent, Dynamic instability

1. Introduction

Contrary to the mainstream equilibrium consideration, when economy is persistently and unpredictably fluctuating, alternative (nonlinear) approaches can better describe the market functioning. Real economic systems are composed of interacting, heterogenous components able to give rise to complex behaviours, putting into question the simplicity of superposition principle that permits the whole to be equal to the sum of its parts. This "unsterilised" description of the economy explains price formation in terms of heterogeneity in preferences and expectations, irrationality, high levels of risk and information gaps. Under similar circumstances, the market can react brutally, disproportionally to incoming news and exacerbate uncertainty even in the absence of exogenous disturbances [Kyrtsou, 2008; Ashley, 2012].

The seemingly-random characteristics of real markets, analysed above, can be gathered under the umbrella of a single definition: a system is complex when it evolves dynamically and it is subject to nonlinear feedback loops between its heterogeneous parts, producing rich endogenous behaviour. Based on these properties, Kyrtsou [2005] presented the Generalised Mackey-Glass (GMG) model, in which interaction between positive and negative feedback rules is capable of producing neglected nonlinearity. Although

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heterogeneity as a source of nonlinearity has determinant role, allowing feedback mechanisms to evolve can additionally affect the system behaviour.

The implications of combining heterogeneity with nonlinearity, in the interactions among either different components of a system or different systems, can be explicitly understood through the concept of dynamic instability. In mathematics and physics, dynamic instability lies at the core of deterministic chaos, which refers to the divergence of nearby trajectories [Schmidt, 2011], and it is an intriguing property of certain simple and well-defined systems which, despite their rudimentary nature, have the ability to evolve in an unforeseeable manner. In cell physiology, it is described as an assembly-disassembly mechanism of microtubules[1] population that facilitates adaptivity to various cellular needs [Mitchison and Kirschner, 1984; Cassimeris et al., 1988]. Finally, in economics and finance, the concept of dynamic instability is not uniquely defined; it is associated with large eigenvalues of the asset correlation matrix [Marsili et al., 2009; Douady and Kornprobst, 2018], it may be due to the explosion of the return variance-covariance matrix in the stochastic equilibrium of a heterogeneous agent based-model [Anufriev et al., 2012], or it can occur in nonlinear models of monetary dynamics [Chiarella, 1986].

There is a widespread view among economists that the complex nature of interconnectivity between the real economy and the financial markets was the root cause of the 2007-2009 crisis. The presence of extreme negative events, momentum behaviour and strong cross-market linkages enhanced contagion and aggravated systemic risk. The spread of contagion depends substantially on the pattern of interdependence between variables [Allen and Gale, 2000]. Minsky's Financial Instability Hypothesis [Minsky, 1986, 1993], according to which instability emerges endogenously, is tightly linked to this new economic paradigm. Economic systems are complex environments and as such they have to defend all the characteristics their nature implies [Haldane, 2013]. The fact that they are composed of heterogeneous sub-systems, co-moving nonlinearly in time, determines a broader notion of dynamic instability.

In the aim to further explore the concept of dynamic instability, we use the GMG model and alter the intensity of the aperiodic perturbation, while raising progressively the degree of nonlinearity. Through numerical simulations, we perform both visual and computational analysis of the artificial time series. These procedures constitute a necessary step for the detection of hidden patterns, since dynamic systems often behave in counterintuitive way due to complex feedback interactions [Kunsch, 2006; Boeing, 2016].

Among the various tools available in nonlinear time series analysis, the Largest Lyapunov Exponent (LLE) can measure the exponential divergence of nearby trajectories and thus accurately quantify the impact of nonlinear feedback mechanisms into the stability of the system. According to O' Gorman et al. [2009], the Lyapunov exponent quantifies the dynamic instability that converts "microscopic" irregularity that is present initially, into "macroscopic" irregularity manifest over time. Our findings show that the obtained LLE values mark divergence depending on the parameter setting of the feedback terms.

The remainder of the paper is organised as follows. In Section 2, we present the GMG model and analyse its fundamental properties. We also report and discuss the results of the simulation experiment. Finally, Section 3 concludes the paper.

2. Model and Simulation Experiment

Based on the GMG model of Kyrtsou [2005], we consider the two-dimensional discrete dynamical system S, composed of two variables X and Y with different types of nonlinearity, that may represent real patterns observed in economic and financial variables, i.e. stock indexes, financial assets, macroeconomic indicators etc (see [Tramontana et al., 2009]).

$$S:\begin{cases} X_{n+1} = & f(X_n) \\ Y_{n+1} = & h(Y_n, X_n) = g(Y_n) + f(X_n) \end{cases} \tag{1}$$

[1]Microtubules are protein structures that, along with other polymers, form the cellular cytoskeleton of eukaryotic cells.

where:

$$\text{LG part: } f(X_n) = \beta X_n(1 - X_n) = \beta X_n - \beta X_n^2$$
$$\text{MG part: } g(Y_n) = \frac{\alpha Y_n}{1 + Y_n^c} - \delta Y_n$$

As reported in Dieci et al. [2001] the behaviour of two-dimensional systems, such as S, is affected by the one-dimensional map X_{n+1} acting as the "driving" factor that contributes to the dynamics of the second map Y_{n+1}, which is called "driven". In (1), the dynamics of the driving variable X are generated by the Logistic equation (LG), named as function $f(X_n)$. The variable Y receives information from $g(Y_n)$, a discrete variant of the Mackey-Glass (MG), perturbed by $f(X_n)$. The evolution of (1) is characterised by the parameters β, α, δ, and c. D' Huys et al. [2011], defines α as the coupling strength, while c represents the degree of nonlinearity in a network of MG equations. Further analytical properties of the MG model can be found in Elhassanein [2014]. In our simulation, we set $\alpha = 2.1$, $\delta = 0.05$, $\beta = 3.6$, 3.9 and 4, while c varies from 2 to 10 and 5.000 observations (transient points have been discarded) are generated via the iteration of the system (1). Different combinations of the MG and LG parameters affect asymmetrically the S and thus induce complex time series dynamics. As explained by Kunsch [2006], the properties of a nonlinear system change because some feedback loops become active, while some other remain dormant.

The dynamic behaviour of the LG component is determined by shifts between positive and negative feedback loops reflected in the βX_n and βX_n^2 terms. It is the nonlinear term that competes with the linear one to stabilise the series [Antoniou et al., 1997]. However, shifting loop dominance may also appear, cancelling out this self-regulating property. As we adjust the β parameter upwards, the map X presents aperiodic fluctuations, while its autocorrelation (ACF) swings from persistent to exponentially decaying and zero values, respectively (Figure 1).

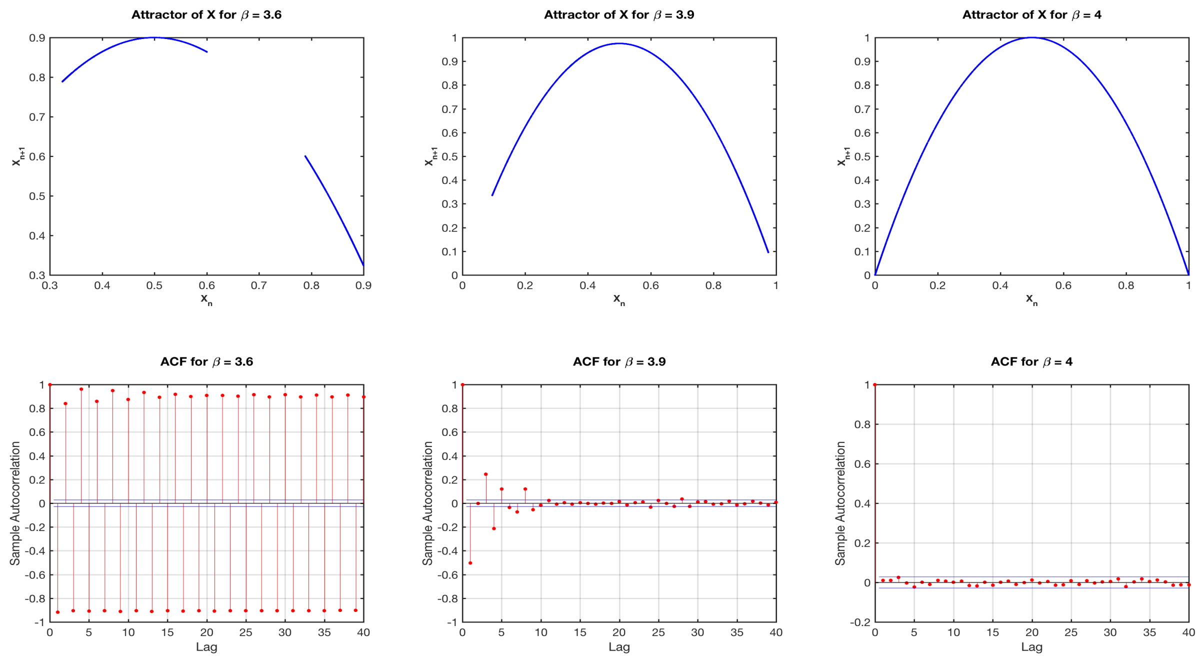

Fig. 1. Attractors and ACF plots for different β values of the Logistic equation

The amplification of dynamics in Y is favoured when in the MG component, the condition $\alpha > \delta$ is satisfied. In this case, positive feedback prevails. Kyrtsou and Malliaris [2009] offer a behavioural interpretation of the c parameter. In financial time series, varying c in the denominator of $\frac{Y_n}{1+Y_n^c}$ accounts for the amplitude of returns, so that present returns are appropriately weighted and affect the future state of the dependent variable in the second map Y. Increasing c updates, through the α parameter, the function $g(\cdot)$ under the driving of $f(\cdot)$.

Having described the construction of (1), we proceed with the examination of the dynamics of nonlinearity. To this end, we computationally investigate how the system stability is affected by the interaction of its individual components. In particular, we first intervene in the behaviour of the map X by setting β equal to 3.6, 3.9 and 4. In response to changes in the $f(\cdot)$ dependence structure, we increase the degree of nonlinearity (c) of the MG part ($g(\cdot)$), from 2 to 10, in the aim to enrich the dynamics of Y. To facilitate the discussion that follows, we consider that the variable X denotes the stock exchange A that acts as a signal (driving) for the stock exchange B, represented by the driven variable Y.

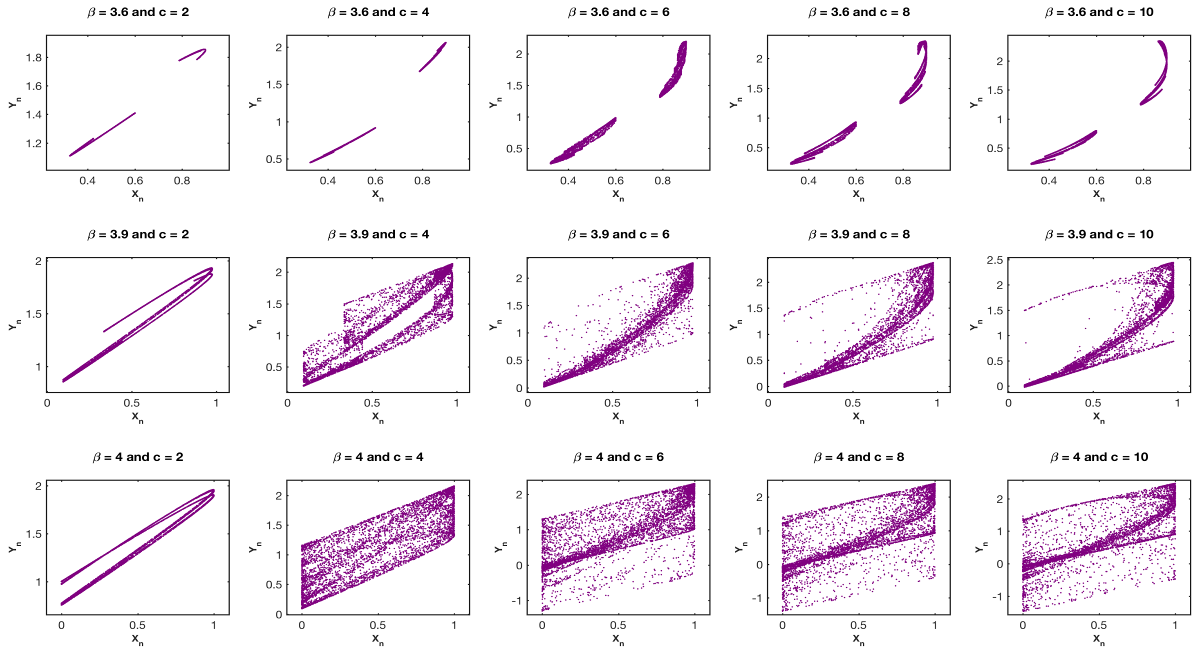

Fig. 2. System attractors for various β and c values

In Figure 2, we plot the evolution of S on the (X, Y) plane for $\alpha = 2.1$, $\delta = 0.05$, and c ranging from 2 to 10, as β equals successively 3.6, 3.9, and 4. These plots demonstrate the dynamic interdependence between the maps X and Y. Moving from the top left to the bottom right attractor, we can identify that the system is fundamentally altered in the view of small variations for the parameters c and β. As c increases (row-wise direction), the attractive region becomes more dense. This finding suggests that c in the function $g(\cdot)$, amplifies the signal received from the X variable and makes the attractor to visit more extended regions than those imposed by the bounded deterministic form of X. On the other hand, as β increases from 3.6 to 4 (column-wise direction) the shape of the attractor is completely changed and the system undergoes a transition from a two-piece attractor to a more compact structure. Therefore, it seems that the driving variable X determines the overall dynamics of the system S, while the driven variable Y is responsible for its expansion. In other words, when c and β get their maximum values, the increased heterogeneity affects significantly the relationship between X and Y. Linking this result with the notion of contagion in real financial markets, we can conclude that when the signal from the stock market A is more aperiodic, then this effect is accelerated in the stock market B.

The (X, Y) plane illustrates the way that the stock markets A and B are dynamically connected. In the case that A and B do not share a common mechanism, the (X, Y) plot would form a cloud of points because the markets are unrelated. If the relationship between A and B follows a deterministic rule, then the (X, Y) plot will depict a non-random structure (i.e. attractor). The shape of this attractor is determined by the blending of the individual properties of the maps X and Y. In our case, the interplay of heterogeneity and nonlinearity leads to the objects reported in Figure 2. In economic terms, this behaviour relates to the existence of an active information flow between the stock markets A and B or to cross-market spillover effects.

Nevertheless, if our goal is to study the data generating mechanism that lies behind the stock market B, assuming that other unobservable factors (such as the stock market A) influence its performance, then a different representation of dynamics is needed. In Figure 3, we report the projections of the attractors on the (Y_n, Y_{n+1}) plane. Visualisation of the function $h(Y_n, X_n)$ outlines the impact of heterogeneous feedback dynamics. As β moves upwards, the shape of the attractor becomes more sensitive to high values of the c parameter. Thus, the strength of nonlinearity (quantified by c) in the stock market B determines its resilience to incoming signals.

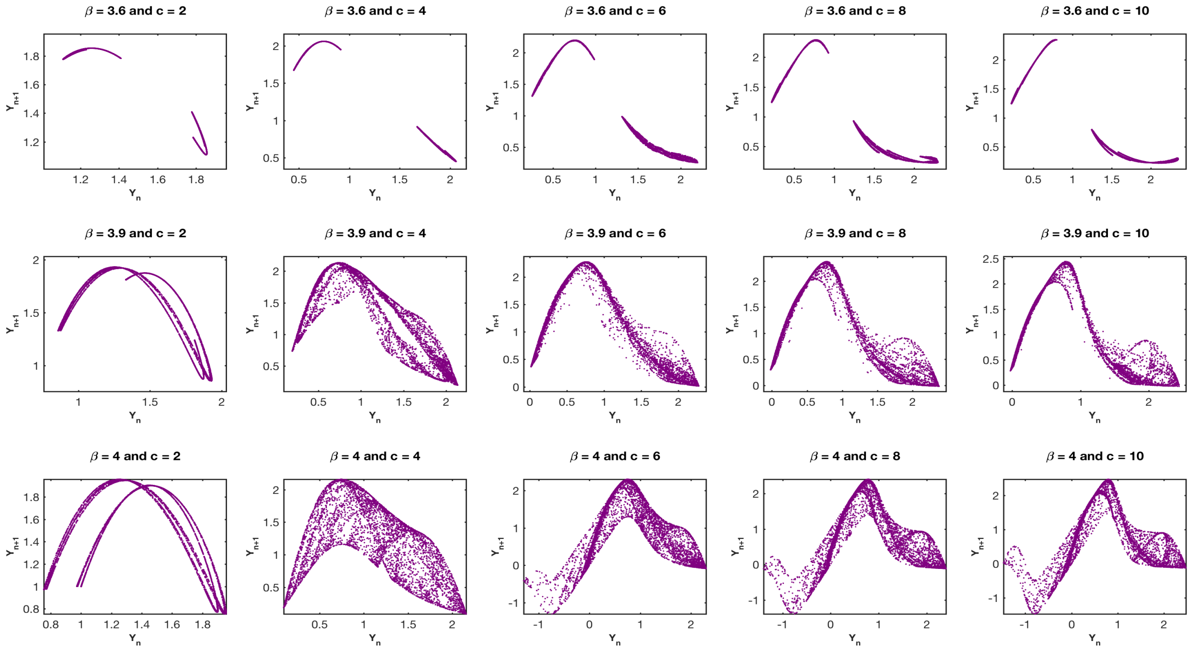

Fig. 3. Attractors of Y for various β and c values

To refine this resilience effect, we compare Figure 1 and 4, and reach two interesting conclusions regarding the action of the feedback mechanism of the MG component. First, as shown by the attractors of Y for $c = 10$ and $\beta = 3.6, 3.9$ and 4 in Figure 4, the significance of the LG part in Y lowers as β increases, while the MG contribution becomes more evident. Moreover, the prevalence of MG is confirmed by the form of the autocorrelation function. That is, while for $\beta = 4$ the signal from the variable X is aperiodic, the MG in Y transforms it into positive short-term autocorrelations. This means that as β progressively gets high values (lower persistence in function $f(\cdot)$), the market-specific dynamics of the stock exchange B absorb the traces left by the exogenous market forcing.

The fact that we are able to group attractors on the basis of the statistical dependence of function $f(\cdot)$, built to affect Y dynamics as a disturbance term, can lead to insightful implications. The nature of persistence in the autocorrelation of a time series has been related to the informational efficiency of the stock markets as well as with the profile of traders they actively affect prices. Short-term autocorrelations often refer to speculative trading, while zero autocorrelations may be the result of the joint presence of heterogeneous investors [Farmer and Joshi, 2002]. In our simulation experiment, it is worth noticing that when the driving variable X exhibits short-term dependence ($\beta = 3.9$) and the degree of nonlinearity reaches high values ($c \geq 6$), the resulting attractor converges to a compact form resembling to the Cinderella's heel.

To get quantitative evidence about the impact of nonlinearity under different forms of aperiodic disturbances, we measure the attractors' instability by calculating the Largest Lyapunov Exponent (LLE), as described in BenSaida and Litimi [2013] and BenSaida [2015]. This test utilises a neural network algorithm

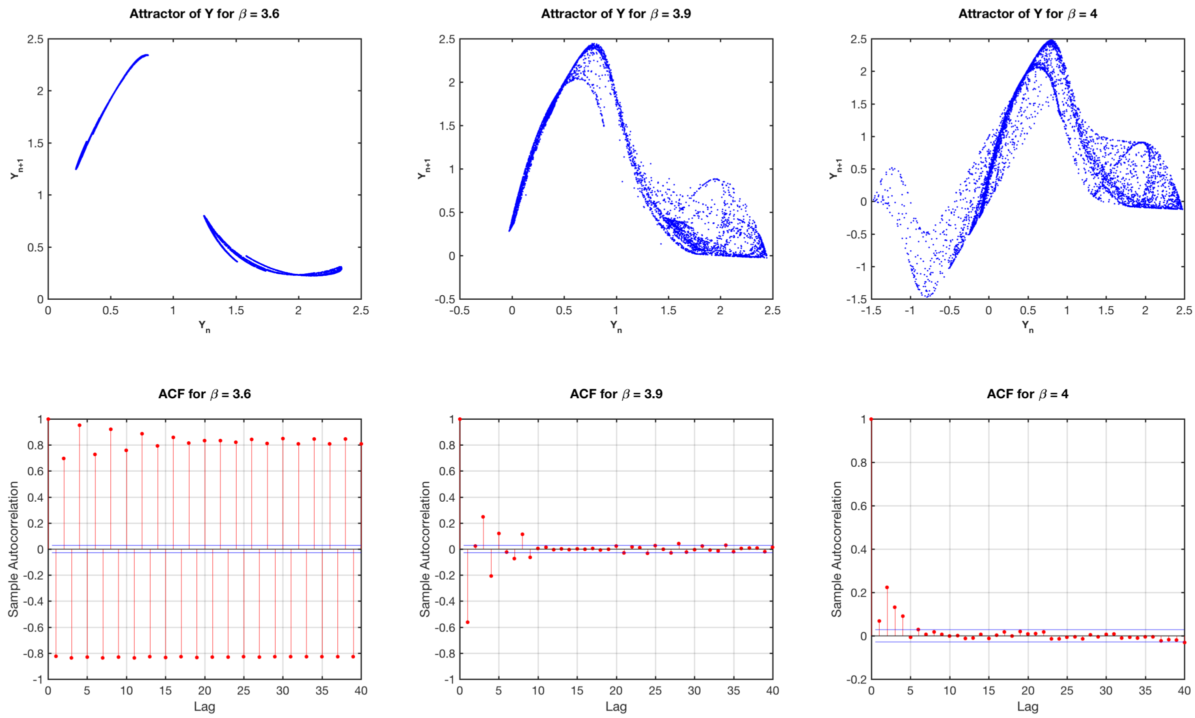

Fig. 4. Attractors and ACF plots for$c = 10$ and different β values of the Y map

to reconstruct adaptively a functional form $\mathcal{F}$ that produces the time series given as input. The estimate of the dominant Lyapunov exponent is the logarithm of the largest eigenvalue of the Jacobian matrix of the function $\mathcal{F}$. Besides the calculation of the LLE (λ), the authors test the null hypothesis H_0 that $\lambda \geq 0$ (presence of chaotic dynamics) against the alternative H_A that $\lambda < 0$ (absence of chaos), at 5% significance level. The more positive the exponent, the greater the instability.

In this regard, our computational study consists of: i) estimating the LLEs on both maps X and Y in their full path and ii) then checking for the temporal variation of LLEs in rolling samples. Thereby, we can evaluate how the different forms of the dependence structure of X are reflected in the build up of the Y and provide evidence about dynamic instability in the driven system under different levels of nonlinearity.

Results are reported in Table 2, together with the accepted hypotheses. As expected, in the case of X, the estimates show that as β increases all LLEs increase as well (Panel A). Regarding Y, the evolution of the LLEs depends crucially on the intensity of nonlinearity c in the MG as well as on the form of autocorrelation of the disturbance term LG (Panel B).

To investigate the triggering factors of dynamic instability, we first consider changes in the c parameter. On this basis, when LG obeys long-range autocorrelations ($\beta = 3.6$), the LLEs deviate aperiodically at about 5.99% from their mean value 0.077. In contrast, if LG exhibits short-range autocorrelations ($\beta = 3.9$), we observe that below a certain threshold of c ($c < 6$) there is a clear tendency of LLEs to rise, sign of amplified instability. Above that threshold, LLE evolution reverses and gets lower values. However, for $\beta = 4$, where LG presents no significant autocorrelations, we detect more volatile LLEs around a higher mean value equal to 0.4190, having a standard deviation of 6.57%. The previous analysis underlines that only when the signal coming from the stock market A presents short-term memory, then strengthening nonlinearity in the driven market B up to a level ($c < 6$) leads to steadily increased LLEs. On the other hand, we put emphasis on the form of autocorrelation for the driving market A. As the disturbance evoked by A becomes more aperiodic, dynamic instability is amplified only when the driven stock market B reaches regions of high nonlinearity ($c \geq 7$). This result highlights the asymmetric contribution of the individual dynamics of function $h(Y_n, X_n)$.

Next, we evaluate the sensitivity of LLEs when new information is added to the initial sample of 5.000 points. We calculate the temporal LLEs of Y as a function of time LLE(t), where $t = 0, .., 10$ represents

Table 1. LLEs of the maps X and Y for various c and β values

Panel A		**Panel B**			
X			Y		
			$\beta = 3.6$	$\beta = 3.9$	$\beta = 4$
$\beta = 3.6$	0.178873	**c = 2**	0.169664 H_0	-1.204756 H_A	-1.204756 H_A
		c = 3	0.028859 H_0	-0.445800 H_A	0.338209 H_0
	H_0	**c = 4**	0.121849 H_0	0.325356 H_0	0.337797 H_0
$\beta = 3.9$	0.526386	**c = 5**	0.073626 H_0	0.522862 H_0	0.499198 H_0
		c = 6	0.047611 H_0	0.513989 H_0	0.460971 H_0
	H_0	**c = 7**	0.032989 H_0	0.388735 H_0	0.447630 H_0
$\beta = 4$	0.686008	**c = 8**	0.077432 H_0	0.238373 H_0	0.449818 H_0
		c = 9	0.000000 H_0	0.267703 H_0	0.468357 H_0
	H_0	**c = 10**	0.150680 H_0	0.207860 H_0	0.350199 H_0

the number of windows. We set the window size equal to 5.000 points and proceed ahead with a step shift of 500 data points. Figure 5 shows the computed LLE values in rolling windows. For each β value, we plot LLE(t) corresponding to different levels of nonlinearity. It turns out that, visiting progressively updated regions of the attractor renders the largest Lyapunov exponent volatile and sensitive to time.

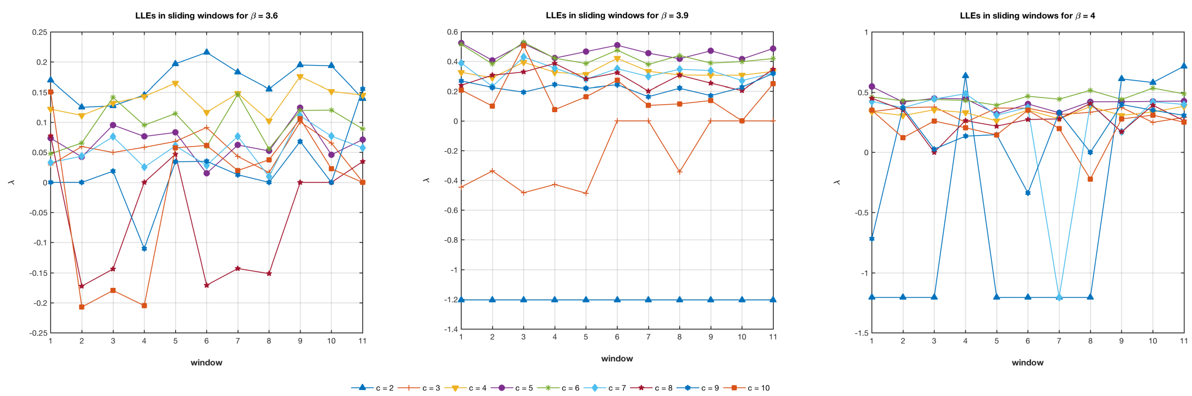

Fig. 5. LLEs of Y in sliding windows for various β and c values.

Table 2 reports the standard deviation as a sensitivity factor of the obtained rolling LLE(t), for given c and β parameters. As anticipated, the sensitivity of LLEs is significant when the dynamics of the driving market are noisy chaotic ($\beta = 4$) and the strength of nonlinearity is above 7. In the context of financial markets, this conclusion concretises the fact that the arrival of random information does not necessary imply market efficiency. Moreover, for all β values, when $c = 10$ highest deviations are also observed. Finally, an interesting finding is revealed for high values of the sensitivity factor where abrupt transitions occur from chaotic to no-chaotic LLE values and vice versa (Figure 5).

In our experiment, dynamic instability describes rising LLEs as a result of varying c and β in the interacting MG and LG components of the system S. The Cinderella attractor constitutes an appealing representation of this computational finding. The asymmetric contribution of functions $f(\cdot)$ and $g(\cdot)$ to the overall dynamics, determines the degree of heterogeneity. For high values of c, above a specific threshold,

Table 2. Standard deviations of LLE values in rolling windows

	$\beta = 3.6$	$\beta = 3.9$	$\beta = 4$
c = 2	0.031179	0.000000	0.928220
c = 3	0.029874	0.224911	0.048563
c = 4	0.022818	0.039628	0.038584
c = 5	0.029145	0.043238	0.059960
c = 6	0.035197	0.053705	0.041120
c = 7	0.029673	0.056326	0.484943
c = 8	0.098466	0.059042	0.125550
c = 9	0.062957	0.043652	0.346569
c = 10	0.125887	0.134841	0.160297

the unstable behaviour reverses and LLE values decline. This finding provides supportive evidence about the dual role of heterogeneity. As nonlinearity strengthens and heterogeneity increases, prices may either stabilise or destabilise [Naimzada and Ricchiuti, 2014; Hommes and in 't Veld, 2017].

3. Conclusions

Inspired by the intricate concept of interaction in mathematics and economics, we studied the dynamics of nonlinear feedbacks in the Kyrtsou [2005] model. Through numerical simulations, we provided evidence about the different sources of dynamic instability, challenging the argument that increasing heterogeneity and nonlinearity relates to destabilising behaviour. Therefore, the interaction among nonlinear feedback mechanisms cannot lead to a univocal interpretation of the system complexity. Computationally, this result is captured by either steadily increasing or fluctuating LLE values, depending on the parameter settings.

In terms of financial implications, we have demonstrated that the effect of the driving stock market A on the performance of the driven stock market B depends substantially on the nature of the inherent structure of both markets. When the underlying dynamics are nonlinear, then the form of the signal transmitted by the driving market is detrimental to the transformation that it is effectuated in the driven market. More specifically, contrary to the traditional view of financial market efficiency, it appears that in a market governed by nonlinear dynamics random information can indeed affect considerably intrinsic market characteristics (i.e. memory) and amplify instability.

Ongoing joint work on simulated data aims to advance the present framework by adding coupled terms and other types of delayed feedback mechanisms so as to better approximate rich statistical phenomena as such observed in real financial and economic time series.

Acknowledgments

Authors would like to thank Professors Anastasios Malliaris of Loyola University Chicago and Dimitris Kugiumtzis of Aristotle University of Thessaloniki, for their helpful comments.

References

Allen, F. and Gale, D. (2000). Financial Contagion. *Journal of Political Economy*, 108(1):1–33.

Antoniou, A., Ergul, N., and Holmes, P. (1997). Market Efficiency, Thin Trading and Non-linear Behaviour: Evidence from an Emerging Market. *European Financial Management*, 3(2):175–190.

Anufriev, M., Bottazzi, G., Marsili, M., and Pin, P. (2012). Excess covariance and dynamic instability in a multi-asset model. *Journal of Economic Dynamics and Control*, 36(8):1142–1161.

Ashley, R. (2012). On the origins of conditional heteroscedasticity in time series. *Korean Economic Review*, 28(1):5–25.

BenSaida, A. (2015). A practical test for noisy chaotic dynamics. *SoftwareX*, 3-4:1–5.

BenSaida, A. and Litimi, H. (2013). High level chaos in the exchange and index markets. *Chaos, Solitons & Fractals*, 54:90–95.

Boeing, G. (2016). Visual Analysis of Nonlinear Dynamical Systems: Chaos, Fractals, Self-Similarity and the Limits of Prediction. *Systems*, 4(4):37.

Cassimeris, L., Pryer, N. K., and Salmon, E. D. (1988). Real-time observations of microtubule dynamic instability in living cells. *The Journal of Cell Biology*, 107(6):2223–2231.

Chiarella, C. (1986). Perfect foresight models and the dynamic instability problem from a higher viewpoint. *Economic Modelling*, 3(4):283–292.

D' Huys, O., Fischer, I., Danckaert, J., and Vicente, R. (2011). Role of delay for the symmetry in the dynamics of networks. *Physical Review E*, 83(4):046223.

Dieci, R., Bischi, G.-I., and Gardini, L. (2001). From bi-stability to chaotic oscillations in a macroeconomic model. *Chaos, Solitons & Fractals*, 12(5):805–822.

Douady, R. and Kornprobst, A. (2018). An empirical approach to financial crisis indicators based on random matrices. *International Journal of Theoretical and Applied Finance*, 21(03):1850022.

Elhassanein, A. (2014). Complex dynamics of a forced discretized version of the Mackey-Glass delay differential equation. *Discrete and Continuous Dynamical Systems - Series B*, 20(1):93–105.

Farmer, D. J. and Joshi, S. (2002). The price dynamics of common trading strategies. *Journal of Economic Behavior & Organization*, 49(2):149–171.

Haldane, A. G. (2013). Rethinking the financial network. In Jansen, S. A., Schroter, E., and Stehr, N., editors, *Fragile Stabilität stabile Fragilität*, pages 243–278. Springer Fachmedien Wiesbaden, Wiesbaden.

Hommes, C. H. and in 't Veld, D. (2017). Booms, busts and behavioural heterogeneity in stock prices. *Journal of Economic Dynamics and Control*, 80:101–124.

Kunsch, P. L. (2006). Limits to Success. The Iron Law of Verhulst. In Ausloos, M. and Dirickx, M., editors, *The Logistic Map and the Route to Chaos*, chapter 3, pages 29–51. Springer-Verlag, Berlin/Heidelberg.

Kyrtsou, C. (2005). Evidence for neglected linearity in noisy chaotic models. *International Journal of Bifurcation and Chaos*, 15(10):3391–3394.

Kyrtsou, C. (2008). Re-examining the sources of heteroskedasticity: The paradigm of noisy chaotic models. *Physica A: Statistical Mechanics and its Applications*, 387(27):6785–6789.

Kyrtsou, C. and Malliaris, A. G. (2009). The impact of information signals on market prices when agents have non-linear trading rules. *Economic Modelling*, 26(1):167–176.

Marsili, M., Raffaelli, G., and Ponsot, B. (2009). Dynamic instability in generic model of multi-assets markets. *Journal of Economic Dynamics and Control*, 33(5):1170–1181.

Minsky, H. P. (1986). Stabilizing an unstable economy. *Hyman P. Minsky Archive*, 144.

Minsky, H. P. (1993). The Financial Instability Hypothesis. In Arestis, P. and Sawyer, M., editors, *Handbook of Radical Political Economy*. Edward Elgar.

Mitchison, T. J. and Kirschner, M. (1984). Dynamic instability of microtubule growth. *Nature*, 312(5991):237–242.

Naimzada, A. K. and Ricchiuti, G. (2014). Complexity with Heterogeneous Fundamentalists and a Multiplicative Price Mechanism. *Economic Notes*, 43(3):233–247.

O' Gorman, D. E., White, J. A., and Shera, C. A. (2009). Dynamical Instability Determines the Effect of Ongoing Noise on Neural Firing. *Journal of the Association for Research in Otolaryngology*, 10(2):251–

267.

Schmidt, J. C. (2011). Challenged by Instability and Complexity. In *Philosophy of Complex Systems*, volume 10, pages 223–254. Elsevier.

Tramontana, F., Gardini, L., Dieci, R., and Westerhoff, F. (2009). The Emergence of Bull and Bear Dynamics in a Nonlinear Model of Interacting Markets. *Discrete Dynamics in Nature and Society*, 2009.