Inventory decisions for a finite horizon problem with product substitution options and time varying demand

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Abstract

The inventory control of substitutable products has been recognized as a worth studying problem in operations management literature. Product substitution provides flexibility in supply chain management and enhances the quick response ability of production control. In this paper, we study a finite horizon inventory control problem for two substitutable products, which are ordered jointly in each replenishment epoch. Demands for the products are assumed as time–varying. In case of a stock–out for one of the products, its demand is satisfied by using the stock of the other product. The determination of the optimal ordering schedule, for both products that minimizes the total cost over a finite planning horizon, is derived. Numerical examples along with sensitivity analysis are also presented.

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1. Introduction

It is well known that effective supply management concerns the ability to match production to demand in order to deal with uncertainty and variability. So, flexibility tools are important in supply chain members (manufacturers, retailers and customers) permitting adaptability and timely responses (Iravani et al. [1]). These flexibility tools help improve product's availability, making a positive impact on customer loyalty and the business performance of retailers and their suppliers/manufacturers, and can be exemplified by product substitution (Trautrims et al. [2]).

Substitution is the practice of using an alternative product in order to meet the demand for another product. This practice has recently gained considerable attention in the operations management literature because it affects the effectiveness of companies decisions with material/product planning, pricing and control (Shin et al. [3]). In general, substitution can take place at the end product level or at the component level and according to Tang and Yin [4], there are three substitution mechanisms: inventory-based (stock-out), assortment-based and price-based substitution. Inventory-based substitution is observed in stock-out situations where it is common practice that consumers, who intend to purchase a particular product, which is not available in the store, are willing to buy a substitute product rather than visit a different shop. A survey mentions that 12%-18% of consumers said that they would not buy a product on a shopping trip if their favorite brand-size was not available; the rest indicated that they would be willing to buy another size of the same brand, or switch brands (Anupindi et al. [5]). In assortment–based substitution the customer chooses a substitute, triggered by the fact that the substitute is newly added in the assortment. In price-based substitution the customers substitution behavior is driven by a change in the relative price of substitutable product.

Another major classification has been made depending on whether the substitution is driven by the consumer (consumer driven substitution) or by the supplier (supplier/manufacturer driven substitution). In consumer–driven substitution, the consumer decides whether he will purchase a substitute or leave empty handed if his basic option product is out of stock (Mahajan and van Ryzin [6]), while in supplier-driven substitution, the supplier decides whether to use a substitute product to meet the demand (Tan and Karabati [7]). Another well–recognized classification is based on whether one– or two–way substitution is used (Stavrulaki [8]). One–way substitution assumes that products can be ordered based on an attribute such as quality, or speed of service. So, products with higher levels of the attribute can substitute for products with lower level of the attribute. On the other hand, two–way substitutability enables consumers to substitute among products within the same category.

From the previous discussion, someone can see that effective retailers inventory control decisions are required by considering substitution among the products. McGillivray and Silver [9] were the first to study substitu-tion in inventory models. They assumed products with identical cost and a fixed substitution probability. They calculated the optimal order quantity using simulation and heuristics methods. Parlar and Goyal [10] studied the same problem and derived optimality conditions for the replenishment pol-icy. Drezner et al. [11] considered a two substitutable products EOQ type model and derived closed–form formulas for the optimal order quantities. They also compared the cases of full substitution and no substitution. Rajaram and Tang [12] proposed a substitutable multiple–product inventory model where they examined the effect of the level of demand correlation and the degree of substitution on the ordering policy. Hsieh and Wu [13] considered a supply chain model with two suppliers, who sell substitutable products through a common retailer. Gurler and Yilmaz [14] studied a two level supply chain model for two substitutable products. They showed the positive impact of substitution on increasing the profit and highlighted the importance of integrating this flexibility tool into the supply chain coordination issues. Stavrulaki [8] studied the joint effect of demand stimulation and product substitution on replenishment policies. Lu et al. [15] demonstrated the product substitution scheme under the multiple sourcing as a risk reduction in the supply chain. Ye [16] considered the competitive in-ventory issues under simultaneously horizontal (inter–brand) and vertical (intra–brand) substitution and proved the existence of equilibrium. Zou and Sun [17] modeled and solved optimally an assemble–to–order inventory system with component and product substitution. Deflem and Van Nieuwen-huyse [18] studied the problem of stocking two products with substitution in both single period and infinite planning horizon. Recently, Salameh et al.[19] studied the two product joint replenishment model with substitution in an economic order quantity framework, as in [11], but allowing for partial substitution when stock–out occurs in one of the products. Maddah et al.[20] extended the model in [19] to multiple products. Krommyda et al. [21] studied a two substitutable products inventory problem under partial substi-tution where the demand of each product is a deterministic function of both stock levels. Bournetas and Kanavetas [22] proposed a joint replenishment model with two products, limited storage capacity and partial substitution of products. Ghoniem and Maddah [23] developed an integrated model for optimizing intertwined retail decisions on assortment, pricing, and inventory for fast-moving consumer goods that are characterized by deterministic demand patterns. Chen et al. [24] examined the optimal inventory policy of a system with product substitution and service level requirements. Interested readers on inventory and production models with product substitution may consult the recent review papers of Lang [25] and Shin et al. [3].

Most of the above surveyed works adopted an infinite planning horizon with constant demand rate. The assumption that the demand is known in advance is applicable if supply contracts are signed ahead of time designating deliveries for the next periods (see Bramel and Simchi-Levi [26]). When the demand rate varies with time the use of the demand information over a fi-nite planning period is required. It is also known that inventory models with time varying demand cover a broad range of practical situations including multi-echelon assembly operations, production to contract, products with seasonal demand etc (see Silver et al. [27]). The linearly time–varying de-mand rate is treated in Resh et al. [28] and Donaldson [29], while Barbosa and Friedman [30] examined inventory models with power form of time de-mand functions. The general time–varying demand lot sizing model was examined in Henery [31] and the same author focused on non–monotonic demand patterns in the special case of cyclic demands in [32]. Recently, Massonnet et al. [33] proposed some approximation algorithms for deter-ministic continuous–review inventory lot–sizing problem with time–varying demand.

The present paper aims to address inventory replenishment decisions for two substitutable products under the inventory-based substitution mechanism. In particular, we examine an inventory control model for two products, which are ordered jointly in every replenishment cycle and have different demand rates depending on time. In case of a stock–out for one of the products, its demand is satisfied by using the stock of the other product. Product substitution affects the inventory control policy as it is not necessary to plan sufficient inventory for each product if the demand is totally independent since the inventory of the substitutable product can be used to satisfy the unmet demand and to avoid lost sales due to the shortages of a particular product (see Song [34]). The mathematical formulation considers general demand functions of time, which permit modelling various market circumstances during the finite planing horizon, while the substitution assumptions of the Drezner et al. [11] model are adopted. The inventory ordering schedule that minimizes the total cost over a finite planning horizon is derived and this is shown to exist uniquely under some technical conditions. The proposed model is applicable to products, where only one of the products substitutes for the other (known as "one-way substitution"). One-way substitution occurs in many settings such as semiconductor chips where a faster processor can be substituted for a slower processor (Hsu and Bassok, [35]). Also, this model could be used in a supplier-driven substitution framework where the supplier uses dual sourcing to meet the demand. In addition, the proposed model is applicable for products where unmet demand of one product is converted to the other, like fast moving consumer goods; nondurable products and grocery items. However, in this case the solution procedure proposed in the present study should be applied 2^n times, for fixed number of orders n during the planning horizon while the role of products labeled as "1" and "2" should be interchanged. However, when n is a decision variable then the computational effort may be prohibitive.

The remainder of this paper is organized as follows. In section 2, we provide the assumptions for the proposed inventory model and the notation to be used throughout the paper. In Section 3, the mathematical model is developed. Details of the optimal replenishment policy are also presented. Numerical examples are provided for illustration in Section 4 and finally a summary and conclusions are found in Section 5. Proofs of the main technical results are contained in the appendices.

2. Assumptions and Notations

The mathematical model is developed under the following assumptions:

1. A two product inventory system is considered over a known and finite planning horizon, $H > 0$.
2. The demand rate for product i ($i = 1, 2$) at time t is denoted by the function $D_i(t) : [0, H] \to \mathbb{R}_+$. It is assumed $D_i(t) \in C^1([0, H])$, where $C^1([0, H])$ is the space of differentiable functions on the interval $[0, H]$.
3. During a demand cycle, both products face their own demand. One of the products, say product 2, exhausts first (this happens in each cycle due to, for example better color or flavor, etc) and product 1 is used to satisfy its demand (say sequence $2 \mapsto 1$). The cycle starts with an instant replenishment of both products and ends when the inventory of product 1 reaches level zero (the results for the sequence $1 \mapsto 2$ is achievable simply by changing indices 1 and 2).
4. Shortages are not allowed.

The notations used throughout the paper are listed below:

k	The set-up cost, $k > 0$
$D_i(t)$	The demand for product i at time t, $i = 1, 2$.
c_i	The holding costs for product i, ($c_i > 0$) $i = 1, 2$, with $c_1 \neq c_2$
c	The transfer cost, $c > 0$.
t_{i-1}	The starting time of cycle i.
t_i	The ending time of cycle i.
s_i	The time of substitution in cycle i.
$I_i(t)$	The inventory level for product i, $i = 1, 2$ at time t.
$'$	The derivative of a univariate function.
∂_x	The partial derivative of a bivariate function with respect to the first variable.
∂_y	The partial derivative of a bivariate function with respect to the second variable.
∂_x^2	The second derivative of a bivariate function with respect to the first variable.
∂_y^2	The second derivative of a bivariate function with respect to the second variable.
$\partial_x \partial_y$	The cross partial derivatives of a bivariate function.
∇	The gradient of a bivariate function.

3. The mathematical formulation of the problem

Figure 1 depicts the inventory of two substitutable products with respect to time in a typical cycle $[t_{i-1}, t_i]$ with time–varying demand rates. At time instant s_i the inventory of product 2 depletes and its demand is subsequently satisfied by inventory of product 1. At time instant t_i the inventory level of product 1 drops to zero and an instant joint replenishment for both products is triggered.

So, on the time interval $[t_{i-1}, s_i]$, the change in inventory level for products 1 and 2 is described by the equations (1) and (2) respectively:

$$\frac{dI_1(t)}{dt} = -D_1(t), \ t_{i-1} \leq t \leq s_i, \tag{1}$$

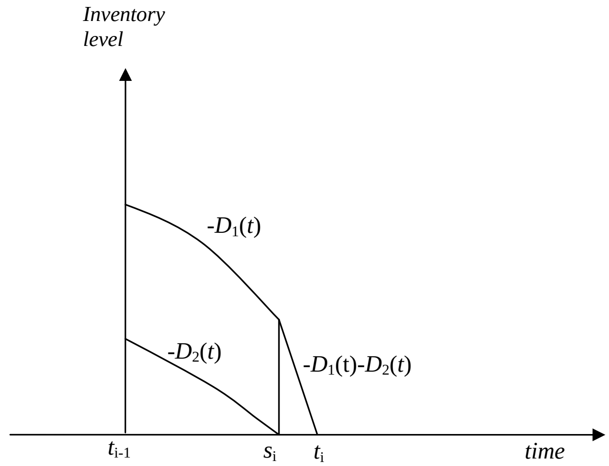

Figure 1: The inventory levels for the two products during the cycle i

with $I_1(s_i^-) = I_1(s_i^+)$. Also,

$$\frac{dI_2(t)}{dt} = -D_2(t),\ t_{i-1} \leq t \leq s_i, \tag{2}$$

with $I_2(s_i) = 0$, while during the time interval $[s_i, t_i]$ the variation of inventory level of product 1 is described as:

$$\frac{dI_1(t)}{dt} = -D_1(t) - D_2(t),\ s_i \leq t \leq t_i, \tag{3}$$

with $I_1(t_i^-) = 0$. The solutions of the above differential equations are:

$$I_1(t) = \int_t^{t_i} D_1(u)du + \int_{s_i}^{t_i} D_2(u)du,\ t_{i-1} \leq t \leq s_i, \tag{4}$$

$$I_2(t) = \int_t^{s_i} D_2(u)du,\ t_{i-1} \leq t \leq s_i, \tag{5}$$

$$I_1(t) = \int_t^{t_i} (D_1(u) + D_2(u))du,\ s_i \leq t \leq t_i. \tag{6}$$

Direct computations show that the holding cost for product 1 is:

$$H_1 := c_1 \left[\int_{t_{i-1}}^{s_i} (t - t_{i-1}) D_1(t) dt + \int_{s_i}^{t_i} (t - t_{i-1}) \{D_1(t) + D_2(t)\} dt \right], \tag{7}$$

the holding cost for product 2 is:

$$H_2 := c_2 \int_{t_{i-1}}^{s_i} (t - t_{i-1}) D_2(t) dt, \tag{8}$$

and the substitution cost is:

$$H_3 := c \int_{s_i}^{t_i} D_2(t)dt. \tag{9}$$

It can be shown that: $H_1 + H_2 + H_3$ is equal to:

$$\int_{t_{i-1}}^{t_i} \{c_1 D_1(t)+c_2 D_2(t)\}(t-t_{i-1})dt+\int_{s_i}^{t_i} \{c-(c_2-c_1)(t-t_{i-1})\}D_2(t)dt. \tag{10}$$

For a given cycle starting at time x and ending at time y with substitution time s $(x \leq s \leq y)$, define

$$\begin{aligned} V(x, s, y) := & \int_x^y \{c_1 D_1(t) + c_2 D_2(t)\}(t - x)dt \\ & + \int_s^y \{c - (c_2 - c_1)(t - x)\} D_2(t)dt. \end{aligned} \tag{11}$$

Consider the mixed integer nonlinear program (MINLP):

$$\begin{array}{ll} \text{P1: min } z = & nk + \sum_{i=1}^{n} V(t_{i-1}, s_i, t_i) \\ \text{subject to:} & 0 = t_0 \leq t_{i-1} \leq \cdots \leq t_n = H \quad . \\ & t_{i-1} \leq s_i \leq t_i, \quad i = 1, \ldots, n. \end{array} \tag{12}$$

The optimal inventory policy reduces to finding n and the vectors (t_{i-1}, s_i, t_i), $i = 1, \cdots, n$ which minimizes z.

It is clear that, when the optimal values for n, t_i, s_i, $i = 1, \ldots, n$ (say n^*, ${t_i}^*$, ${s_i}^*$, $i = 1 \ldots n^*$) are determined, then the optimal ordering quantities for the two products can be determined: $I_1({t_i}^*)$, $i = 1 \ldots n^*$ (ordering quantity for product 1 at replenishment instant ${t_i}^*$), $I_2({t_i}^*)$, $i = 1 \ldots n^*$ ordering quantity for product 2 at replenishment instant ${t_i}^*$).

Note that:

(i) If $s = y$, then this leads to a no substitution case and

$$V(x, s, y) := \int_x^y \{c_1 D_1(t) + c_2 D_2(t)\}(t - x)dt. \tag{13}$$

(ii) If $s = x$, then this case leads to a full substitution case and

$$V(x, s, y) := c_1 \int_x^y (t - x)\{D_1(t) + D_2(t)\}dt + c \int_x^y D_2(t)dt. \quad (14)$$

(iii) If $x < s < y$, then we have partial substitution.

The next result is related to the possibility of occurrences of cases (i)–(iii) above. Let $\tilde{\mathrm{P}}1$ the nonlinear program obtained from $P1$ by fixing the integer variable n.

Theorem 1. *If Problem $\tilde{\mathrm{P}}1$ has a solution $(t^*_{i-1}, s^*_i, t^*_i)$ for $i = 1, \ldots, n$, then*

$$s^*_i = \begin{cases} t^*_i & \textit{if } \gamma > t^*_i - t^*_{i-1} > 0 \ \textit{or } \gamma \leq 0 \\ t^*_{i-1} + \gamma & \textit{if } t^*_{i-1} < t^*_{i-1} + \gamma \leq t^*_i \end{cases}, \quad (15)$$

where

$$\gamma = \frac{c}{c_2 - c_1}. \quad (16)$$

The proof of the theorem is found in Appendix A.

Remark 1.

1. *Theorem 1 shows that at the optimal solution the case of full substitution (case (ii)) does not occur. The same argument shows that it does not also occur in the EOQ model with constant demand. This was pointed out by Drezner et al. [11] and commented by Goyal [36]. We shall comment on this case later on: see Remark 7 below.*
2. *The substitution is possible if $\gamma > 0$ and $c < (c_2 - c_1)(t_i^* - t_i{}^*{}_{-1})$. The first relation implies that $c_2 > c_1$ and consequently it is reasonable the product 2 to be substituted by the product 1, however the second relation requires that the profit from the substitution (in terms of holding cost) should be greater than the transfer cost. Else, no substitution occurs.*

This also is supported by noticing that the second integral term on the right hand-side of (11) can be written as

$$-(c_2 - c_1)\int_s^y (t-s)D_2(t)dt + \{c - (c_2 - c_1)(s-x)\}\int_s^y D_2(t)dt, \quad (17)$$

so when $c_2 - c_1 < 0$ *no substitution (i.e.* $s = y$*) is more profitable.*

3. *The model can be easily modified so that a fraction of the demand for product 2 to be satisfied by product 1. To this end, a cost per unit of lost demand could be taken into account, which can be handled as the transfer cost.*

Guided by the proceeding analysis, in order to solve $P1$, we shall consider separately, the cases $\gamma > 0$ (this is equivalent to $c_2 > c_1$), and $\gamma < 0$ (this is equivalent to $c_2 < c_1$). When $\gamma > 0$ cases (i) and (iii) above may occur and in the second only case (i) occurs.

Let

$$\Omega = \{(x, y) : 0 < x < y \leq H\}. \quad (18)$$

For $j = 1, 2$, and $(x, y) \in \Omega$ define

$$r_j(x, y) = \int_x^y (t-x)D_j(t)dt, \quad (19)$$

$$R_1(x, y) := c_1 r_1(x, y). \quad (20)$$

For $\gamma > 0$, using (10) with (17) and Theorem 1, let

$$R_2(x, y) := c_2 r_2(x, y) - (c_2 - c_1)\begin{cases} r_2(x+\gamma, y), & \text{if } x + \gamma \leq y \\ 0, & \text{otherwise} \end{cases}. \quad (21)$$

For $\gamma < 0$, using (10) with (17) and Theorem 1, set

$$R_2(x, y) := c_2 r_2(x, y). \quad (22)$$

Write

$$R(x, y) := R_1(x, y) + R_2(x, y), \quad (23)$$

3.1. The case $\gamma > 0$.

Using Theorem 1 Problem P1 defined in (12) is equivalent to the following MINLP:

$$\begin{array}{ll} \text{P2. min } z = & nk + \sum_{i=1}^{n} R(t_{i-1}, t_i) \\ \text{subject to:} & 0 = t_0 \leq t_{i-1} \leq \cdots \leq t_n = H \end{array} .$$

where R is given by (23) and the decision variables are: n and t_i, $i = 1, \cdots, n-1$.

Theorems 2 and 3 below present the main results of the paper. Theorem 2 states that for fixed n, P2 has a unique solution under some technical conditions. Theorem 3 proves the convexity of the objective function with respect to n at optimal t_i, $i = 1, \cdots, n-1$. This, as it turned out, is key in obtaining the optimal inventory policy.

Remark 2. *Problem* P2 *as it stands is new in the literature for inventory models with substitutions. If $c_1 = 0$, then* P2 *has some technical similarities with a problem treated in Benkherouf and Gilding [37] for optimal replenishment policies for a finite planning horizon inventory model with deteriorating items and permissible delay in payments. To solve* P2 *we shall draw on ideas from Benkherouf and Gilding ([38] and [37]).*

3.1.1. Preliminaries

The next lemma shows that the functions R_1 and R_2, defined in (20) and (21), respectively are differentiable on the set $\bar{\Omega}$, where $\bar{\Omega}$ is the closure of the set Ω.

Lemma 1. *The functions R_1, and $R_2 \in C^1(\bar{\Omega})$.*

The proof of the Lemma is found in Appendix B. Before we proceed further, the following computations are needed. For $j = 1, 2$

$$\partial_x r_j(x,y) = -\int_x^y D_j(t)dt, \tag{24}$$

$$\partial_y r_j(x,y) = (y-x)D_j(y), \tag{25}$$

$$\partial_x\partial_y r_j(x,y) = -D_j(y), \tag{26}$$

$$\partial_x^2 r_j(x,y) = D_j(x), \tag{27}$$

$$\partial_y^2 r_j(x,y) = D_j(y) + (y-x)D_j'(y). \tag{28}$$

Remark 3. *The function R in (23) is not twice differentiable in $\bar{\Omega}$.*

Indeed, for $j = 1, 2$, the function $r_j(x, y)$ is twice differentiable in $\bar{\Omega}$. If differentiability problems occur then these happen on the line segment $(x, x + \gamma)$. We have by (26)–(28)

$$\partial_x^2 r_2(x, x+\gamma) = D_2(x), \tag{29}$$

$$\partial_y^2 r_2(x, x+\gamma) = D_2(x+\gamma) + \gamma D_2'(x+\gamma), \tag{30}$$

$$\partial_x\partial_y r_2(x, x+\gamma) = -D_2(x+\gamma), \tag{31}$$

For $x+\gamma < y$, $\partial_x^2 r_2(x, x+\gamma) \neq 0$, $\partial_y^2 r_1(x, x+\gamma) \neq 0$, and $\partial_x\partial_y r_1(x+, x+\gamma) \neq 0$. Therefore, R is not twice differentiable by (21).

Our objective next is to solve P2 by appealing to the established methodology in Benkherouf and Gilding [38]. Nonetheless, this methodology requires that R be twice differentiable. This as we shall see will not be a handicap to its application. Special care needs to be taken at the points where the second order differentiability fails to be satisfied.

For an arbitrary function Z, let $(\partial_x Z)(x^+, y)$, $(\partial_y Z)(x, y^+)$ denote the partial derivative of Z with respect to x and y from the right, respectively.

Also, let $(\partial_x Z)(x^-, y)$, $(\partial_y Z)(x, y^-)$ denote the partial derivative of Z with respect to x and y from the left, respectively. A key requirement for the methodology developed in [38] to be applicable is $\partial_x \partial_y R(x, y) < 0$ in Ω. However, the definition of R in (23) with the presence of two demand rate functions (D_1 and D_2) means that $\partial_x \partial_y R_1(x,\ y) < 0$, $\partial_x \partial_y R_2(x^{\pm},\ y) < 0$, and $\partial_x \partial_y R_2(x,\ y^{\pm}) < 0$ in Ω will suffice. Here, $\partial_x \partial_y R_2(x^{\pm},\ y)$ stands for cross partial derivatives on the right and left for the first argument. $\partial_x \partial_y R_2(x,\ y^{\pm})$ refers to the cross partial derivatives on the right and the left for the second argument. The requirement $\partial_x \partial_y R_1(x,\ y) < 0$ is trivial from (26) and the definition of R_1 in (20). For R_2, we shall check the requirement on the sets

$$\Omega_1 = \{(x, y) : 0 < x < y, \text{ and } x + \gamma < y\}, \tag{32}$$

$$\Omega_2 = \{(x, y) : 0 < x < y, \text{ and } x + \gamma > y\}. \tag{33}$$

In Ω_2, the definition of R_2 in (21) and (26) give $\partial_x \partial_y R_2(x, y) = -c_2 D_2(y)$, which is strictly negative. In Ω_1, $\partial_x \partial_y R_2(x, y) = -c_1 D_2(y)$, which is strictly negative. This completes the proof.

As a result we have for $(x, y) \in \Omega$,

$$\partial_x \partial_y R(x^{\pm}, y) < 0, \tag{34}$$

$$\partial_x \partial_y R(x, y^{\pm}) < 0. \tag{35}$$

Expressions (34) and (35) imply that the cost function R is submodular in Ω. For more information on submodular functions and their properties you may consult Topkis [39].

For some functions z and f_j, $(j = 1, 2)$, define the operators

$$\mathcal{L}_x^{(j)} z := \partial_x^2 z + \partial_x \partial_y z + f_j(x) \partial_x z, \tag{36}$$

$$\mathcal{L}_y^{(j)} z := \partial_y^2 z + \partial_x \partial_y z + f_j(y) \partial_y z. \tag{37}$$

Remark 4. *Applications of the operators (36) and (37) in the literature have been for the case* $j = 1$*. The present paper appears to be the first to call for application of the operators beyond* $j = 1$*.*

Lemma 2. *For* $j = 1, 2$*, let*

$$f_j(t) := -\frac{D'_j(t)}{D_j(t)}, \tag{38}$$

then the functions $\mathcal{L}_x^{(j)} R_j$*, and* $\mathcal{L}_y^{(j)} R_j$ *are continuous in* $\bar{\Omega}$*. Moreover, if* f_j *is non-decreasing, then* $\mathcal{L}_x^{(j)} R_j \geq 0$ *and* $\mathcal{L}_y^{(j)} R_j = 0$*, in* $\bar{\Omega}$*.*

The proof of the Lemma is found in Appendix C.

3.1.2. Some basic assumptions

Before we proceed further, we shall make the following assumptions:

Assumptions

(A1) For $j = 1, 2$, $f_1(t) = f_2(t) = f(t)$.

(A2) The function f is non-decreasing in t.

Remark 5.

(i) Assumption (A1)*, implies that* $D_1(t) = AD_2(t)$*, where* A *is some strictly positive constant. This means that the demand of product 1 is proportional to the demand of product 2. This seems to be reasonable since products 1 and 2 are substitutable and from practical point of view is met in the Joint Replenishment Problem (JRP) which refers to coordinated replenishment of a group of items (in the same category) that may be jointly ordered from a single supplier (see in [19]).*

(ii) Assumption (A1) *ensures that Problem* P2 *is solvable using the methodology of Benkherouf and Gilding [38].*

(iii) Assumption (A2) *reduces to the usual condition that the demand functions D_1 and D_2 are log-concave. This assumption is also found in Henery [31], where an economic lot size model is investigated.*

(iv) Under Assumption (A1), *and omitting the superscript notation, we may write*

$$\mathcal{L}_x R = \mathcal{L}_x R_1 + \mathcal{L}_x R_2, \tag{39}$$

$$\mathcal{L}_y R = \mathcal{L}_y R_1 + \mathcal{L}_y R_2. \tag{40}$$

Also, for $(x, y) \in \Omega$ we have

$$\mathcal{L}_x R \geq 0, \quad \mathcal{L}_y R \geq 0, \tag{41}$$

under Assumption (A2).

(v) The possibility of assumptions A1 *and* A2 *fail to hold cannot be ruled out in applications. The implication of this possibility will be discussed later in this section.*

3.1.3. The optimal inventory policy

This subsection contains the details of the optimal inventory policy for the finite horizon inventory model with substitution. As a matter of fact, a procedure for determining the optimal policy fixes initially the integer variable n in P2 resulting in a pure nonlinear programming problem whose solution is given by the following theorem.

Theorem 2. *Under assumptions* A1 *and* A2, *and for fixed n, Problem* P2 *has a unique optimal solution. This solution is obtained by setting*

$$\nabla G_n(t_0, t_1, \cdots, t_n) = 0,$$

where

$$G_n(t_0, t_1, \cdots, t_n) = \sum_{i=1}^{n} R(t_{i-1}, t_i), \tag{42}$$

where $t_0 = 0$, $t_n = H$, and R is given by (23).

The proof of the theorem is found in Appendix D

Remark 6. *Assumption* A1 *ensures that Relation (D9) in the appendix, which is key in the proof of Theorem 2, is true. This relation leads to (D10) and (D11) which are essential in the rest of the proof of the theorem. It would be interesting to know if (D10) and (D11) can be obtained without the need to the present form of (D9). An objective which remains elusive up to the present time.*

The following Theorem is concerned with the determination of the optimal number of replenishment cycles. The proof is omitted and may be found in Benkherouf and Gilding [38] and Denardo et al. [40]. Write

$$g_n := g_n(H) = \min_{t_0, t_1, \ldots, t_n} G_n(t_0, t_1, \ldots, t_n),$$

for the optimal value of $G_n(t_0, t_1, \ldots, t_n)$ for fixed n, where G_n is defined in (42).

Theorem 3. *The optimal number of cycles n^* in Problem* P2 *is given by:*
(i) if $k > g_1 - g_2$, then $n^* = 1$,
(ii) if there exists $N \geq 2$ such that $g_{N-1} - g_N > k > g_N - g_{N+1}$, then $n^* = N$,
(iii) if there exists $N \geq 2$ such that $k = g_N - g_{N+1}$, then $n^* = N$, and $n^* = N + 1$.

Theorem 3 shows that the function g_n is convex in n. This theorem ensures the existence of optimal n (notice that $\lim_{n \longrightarrow \infty} g_n = \infty$).

3.2. The case $\gamma < 0$.

For $j = 1, 2$ define

$$R_j(x, y) = c_j r_j(x, y), \tag{43}$$

where r_j is defined in (19).

The problem of finding an inventory policy is restated with

$$R(x, y) = R_1(x, y) + R_2(x, y), \tag{44}$$

where R_j is defined in (43) for $j = 1, 2$.

The function R given by (44) is smooth is Ω. Therefore, the results for the case $\gamma < 0$ can be recovered under assumptions (A1) and (A2). Details are omitted since the set-up is similar to that for the case $\gamma > 0$. As a matter of fact the search for the optimal inventory policy is simpler than the previous case since the singular points are absent.

Remark 7.

1. *Note that if assumptions (A1)-(A2) fail to hold, Theorem 2 still holds but uniqueness is lost. Theorem 3 remains valid.*
2. *In [36] it was suggested, among other things, that adding a fixed (penalty) cost for substitution in the basic model of Drezner et al. [11] would allow for the possibility for the full substitution case, which is absent in the basic model. This is possible to achieve in the present model. However, the resulting optimization presents a challenging problem since continuity is lost. We suspect that the results developed in [40] would be key in shedding some light on this problem.*
3. *The possibility of $c_1 = c_2$ cannot be ruled out in practice. It is an easy exercise to see that this results in the no substitution case in all cycles, where the optimality is found using the same analysis as in the case*

$\gamma < 0$. Also the same phenomena reappears, if $c = 0$ and $c_2 < c_1$. On the other hand, if $c = 0$ and $c_2 > c_1$, then full substitution is observed. The optimal policy is found using for example the method found in [38].

4. Numerical examples for different demand configurations

In this section we present some examples along with their sensitivity analysis to illustrate the applicability of theorems 2 and 3. In order to examine the effect of time varying demand rate, we consider scenarios with exponential and linear demand rates. In addition, we assume constant demand rates for both product. To this end, the demand rates are approximated by the average demand per unit i.e. $D_1(t) = D_1 = \frac{\int_0^T D_1(t)dt}{T}$, $D_2(t) = D_2 = \frac{\int_0^T D_2(t)dt}{T}$. Moreover, the implication of varying a number of key parameters on the model is examined.

4.1. Exponential decreasing demand

For the first example the following parameters values are used:
$k = 1000$, $c_1 = 3$, $c_2 = 5$, $c = 5$, $H = 5$, $D_1(t) = 80e^{-0.2t}$, $D_2(t) = 60e^{-0.2t}$.

The demand of spare parts for a range of products or agriculture products decreases with time exponentially (see for example Hill et al. [41] and Wee [42]). For the above data the optimal number of orders is $n = 2$ and the optimal cost is 3923.76. The optimal values of t_i, s_i $i = 1, ..., n$ are shown in Table 1. Although $c_1 < c_2$ no substitution occurs in first cycle, probably because of the high ordering cost which affects $t_1 - t_0$ and consequently the relation $c < (c_2 - c_1)(t_1 - t_0)$ does not hold.
Table 2 presents the replenishment policy using constant approximation for

Table 1: The optimal replenishment schedule for Example 1

t_0^*	s_1^*	t_1^*	s_2^*	t_2^*
0	2.134	2.134	4.634	5

the demand rates i.e. $D_1(t) = D_1 = 50.57$, $D_2(t) = D_2 = 37.93$, the corresponding cost is 4133.41. From Table 2, it seems that this approximation affects the replenishment schedule (compare with Table 1). Under constant demand (no time demand variability), there is no product substitution in the optimal replenishment policy. This leads to a cost increase around 5.4%, which, by taking into account the short planning horizon, shows the effect of the demand variability to cost.

Table 2: The optimal replenishment schedule for Example 1 assuming constant demand rate for the two products

t_0^*	s_1^*	t_1^*	s_2^*	t_2^*
0	2.5	2.5	5	5

In order to evaluate the effect of changing parameter values on opti-mal replenishment policy, both on constant and time–varying demand, we perform an sensitivity analysis on all the cost parameters. The results presented in Table 3 summarize the sensitivity analysis with respect to the cost parameters, i, for the data of Example 1, while Table 4 gives the sensitivity analysis for Example 1 by assuming $D_1(t) = D_1$ and $D_2(t) = D_2$. The main results and insights, obtained from Tables 3 and 4, are discussed below.

Table 3: Sensitivity analysis for Example 1

parameter (i)	n^*	optimal policy	Optimal cost
$k = 500$	3	t_0^* = 0, s_1^* = 1.392, t_1^* = 1.392, s_2^* = 3.024, t_2^* = 3.024, s_3^* = 5, t_3^* = 5	2819.13
$k = 750$	2	t_0^* = 0, s_1^* = 2.134, t_1^* = 2.134, s_2^* = 4.634, t_2^* = 5	3423.76
$k = 1250$	2	t_0^* = 0, s_1^* = 2.134, t_1^* = 2.134, s_2^* = 4.634, t_2^* = 5	4423.76
$k = 1500$	1	t_0^* = 0, s_1^* = 2.5, t_1 = 5	4903.12
$c = 2.5$	2	t_0^* = 0, s_1^* = 1.25, t_1^* = 2.138, s_2^* = 3.388, t_2^* = 5	3829.9
$c = 3.75$	2	t_0^* = 0, s_1^* = 1.875, t_1^* = 2.107, s_2^* = 3.982, t_2^* = 5	3900.78
$c = 6.25$	2	t_0^* = 0, s_1^* = 1.875, t_1^* = 2.107, s_2^* = 3.982, t_2^* = 5	3926.53
$c = 7.5$	2	t_0^* = 0, s_1^* = 2.164, t_1^* = 2.164, s_2^* = 5, t_2^* = 5	3926.53
$c_1 = 1.5$	1	t_0^* = 0, s_1^* = 1.428, t_1^* = 5	3140.19
$c_1 = 2.25$	2	t_0^* = 0, s_1^* = 1.818, t_1^* = 2.066, s_2^* = 3.884, t_2^* = 5	3670.69
$c_1 = 3.75$	2	t_0^* = 0, s_1^* = 2.164, t_1^* = 2.164, s_2^* = 5, t_2^* = 5	4140.59
$c_1 = 4.5$	2	t_0^* = 0, s_1^* = 2.164, t_1^* = 2.164, s_2^* = 5, t_2^* = 5	4354.65
$c_2 = 2.5$	2	t_0^* = 0, s_1^* = 2.164, t_1^* = 2.164, s_2^* = 5, t_2^* = 5	3391.39
$c_2 = 3.75$	2	t_0^* = 0, s_1^* = 2.164, t_1^* = 2.164, s_2^* = 5, t_2^* = 5	3658.96
$c_2 = 6.25$	2	t_0^* = 0, s_1^* = 1.538, t_1^* = 2.100, s_2^* = 3.639, t_2^* = 5	4101.47
$c_2 = 7.5$	2	t_0^* = 0, s_1^* = 1.111, t_1^* = 2.117, s_2^* = 3.228, t_2^* = 5	4190.62

Table 4: Sensitivity analysis for Example 1 assuming constant demand rates for two products

parameter (i)	n^*	optimal policy					Optimal cost
		t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	
$k = 500$	2	0	2.5	2.5	5	5	2956.36
$k = 750$	2	0	2.5	2.5	5	5	3633.41
$k = 1250$	2	0	2.5	2.5	5	5	4633.41
$k = 1500$	2	0	2.5	2.5	5	5	5133.41
$c = 2.5$	2	0	1.25	2.5	3.75	5	4014.88
$c = 3.75$	2	0	1.875	2.5	4.375	5	4103.78
$c = 6.25$	2	0	1.875	2.5	4.375	5	3932.54
$c = 7.5$	2	0	2.5	2.5	5	5	4133.41
$c_1 = 1.5$	1	0	1.428	5			3472.04
$c_1 = 2.25$	2	0	1.818	2.5	4.318	5	3847.88
$c_1 = 3.75$	2	0	25	2.5	5	5	4370.45
$c_1 = 4.5$	2	0	2.5	2.5	5	5	4607.5
$c_2 = 2.5$	2	0	2.5	2.5	5	5	3540.79
$c_2 = 3.75$	2	0	2.5	2.5	5	5	3837.1
$c_2 = 6.25$	2	0	1.538	2.5	4.038	5	4315.75
$c_2 = 7.5$	2	0	1.111	2.5	3.611	5	4396.79

4.2. Linear increasing demand

The demand of a new product increases linearly with time when it substitutes an existing product in most electronics, automobiles and seasonal products with short life (San Jose et al. [43]). So, for the second example we assume $D_1(t) = 96 + 6t$ and $D_2(t) = 80 + 5t$. For these data the optimal number of cycles is $n = 3$ and the optimal cost is 6360.06. Table 5 gives the optimal values of t_i, s_i $i = 1, ..., n$.

Table 5: The optimal replenishment schedule for Example 2

${t^*}_0$	${s^*}_1$	${t^*}_1$	${s^*}_2$	${t^*}_2$	${s^*}_3$	${t^*}_3$
0	1.740	1.740	3.403	3.403	5	5

Table 6 presents the optimal replenishment schedule assuming constant demand for the two products (i.e. $D_1(t) = D_1 = 111$ and $D_2(t) = D_2 = 92.5$) and the corresponding optimal cost is 6314.58. According to the op-timal replenishment schedule no substitution occurs although $c_2 > c_1$. An explanation could be the increasing (in time) demand for the two products, so perhaps is more profitable both products to be available.

Table 6: The optimal replenishment schedule for Example 2 assuming constant demand for both products

${t^*}_0$	${s^*}_1$	${t^*}_1$	${s^*}_2$	${t^*}_2$	${s^*}_3$	${t^*}_3$
0	1.666	1.666	3.333	3.333	5	5

Table 7 presents the sensitivity analysis with respect to some important model parameters, i for Example 2, while the respective sensitivity analysis for Example 2 assuming constant approximations for demand rates is presented in Table 8.

The main managerial insights from the above two examples and espe-cially from Tables 3–4 and 7–8, are:

1. From the use of a decreasing in time exponential function and an in-

creasing in time linear function for the description of demand rate (see Table 3 and 7) it seems that the replenishment schedule are affected significantly. In addition, Tables 3 and 4 and Tables 7 and 8 indicate that differences on replenishment schedule and cost are also notable when constant approximation for demand rate is used. So, the necessity of appropriate demand forecast is highlighted as well as the use of the proposed procedure for the determination of the optimal replenishment schedule than a constant approximation for the demand rate. It should be noticed that for decreasing in time demand rates the constant approximation of demand rates leads to overestimation of the total cost while the opposite is happened for increasing in time demand rates.

2. As it has been pointed out when $c_1 > c_2$ no substitution occurs.
3. Although when $c_1 < c_2$ the substitution of the second product is expected, in many cases this does not happen either because of high ordering cost, which leads to short reorder intervals, or because of the high transfer cost, c.
4. The increase in set up cost, k, decreases the number of orders and increase the total cost, as it was expected.
5. The total cost seems to be more sensitive to changes in ordering cost and less sensitive to changes in other costs.
6. The expression $t_i - s_i$ represents the time period in cycle i that product 2 is substituted by product 1. From Table 3 it is observed that $t_i - s_i$ increases in i and this may be due to decreasing demand, the opposite is observed in Table 7.

Table 7: Sensitivity analysis for Example 2

parameter (i)	n^*	optimal policy									Optimal cost
$k = 500$	4	t_0^*	s_1^*	$t1^*$	s_2^*	t_2^*	s_3^*	t_3^*	s_4^*	t_5^*	4510.50
		0	1.314	1.314	2.582	2.582	3.809	3.809	5	5	
$k = 750$	4	t_0^*	s_1^*	t_1^*	t_2^*	t_2^*	s_3^*	t_3^*	s_4^*	t_4^*	5510.50
		0	1.314	1.314	2.582	2.582	3.809	3.809	5	5	
$k = 1250$	3	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*			7110.06
		0	1.740	1.740	3.403	3.403	5	5			
$k = 1500$	3	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*			7860.06
		0	1.740	1.740	3.403	3.403	5	5			
$c = 2.5$	3	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*			6309.85
		0	1.25	1.753	3.003	3.415	4.665	5			
$c = 3.75$	3	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*			6360.06
		0	1.740	1.740	3.403	3.403	5	5			
$c = 6.25$	3	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*			6360.06
		0	1.740	1.740	3.403	3.403	5	5			
$c = 7.5$	3	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*			6360.06
		0	1.740	1.740	3.403	3.403	5	5			
$c_1 = 1.5$	3	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*			5626.33
		0	1.428	1.797	3.226	3.457	4.885	5			
$c_1 = 2.25$	3	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*			6008.42
		0	1.740	1.740	3.226	3.403	5	5			
$c_1 = 3.75$	3	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*			6711.69
		0	1.740	1.740	3.226	3.403	5	5			
$c_1 = 4.5$	4	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*	s_4^*	t_4^*	7035.95
		0	1.314	1.314	2.581	2.581	3.808	3.808	5	5	
$c_2 = 2.5$	3	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*			5383.3
		0	1.740	1.740	3.226	3.403	5	5			
$c_2 = 3.75$	3	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*			5871.68
		0	1.740	1.740	3.226	3.403	5	5			
$c_2 = 6.25$	3	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*			6838.99
		0	1.538	1.771	3.309	3.432	4.970	5			
$c_2 = 7.5$	3	t_0^*	s_1^*	t_1^*	s_2^*	t_2^*	s_3^*	t_3^*			7137.31
		0	1.111	1.763	2.874	3.424	4.535	5			

Table 8: Sensitivity analysis for Example 2 assuming constant demand rates for two products

parameter (i)	n^*	optimal policy	Optimal cost
$k = 500$	4	$t_0^*=0$, $s_1^*=1.25$, $t_1^*=1.25$, $s_2^*=2.5$, $t_2^*=2.5$, $s_3^*=3.75$, $t_3^*=3.75$, $s_4^*=5$, $t_4^*=5$	4485.94
$k = 750$	4	$t_0^*=0$, $s_1^*=1.25$, $t_1^*=1.25$, $s_2^*=2.5$, $t_2^*=2.5$, $s_3^*=3.75$, $t_3^*=3.75$, $s_4^*=5$, $t_4^*=5$	5485.94
$k = 1250$	3	$t_0^*=0$, $s_1^*=1.666$, $t_1^*=1.666$, $s_2^*=3.333$, $t_2^*=3.333$, $s_3^*=5$, $t_3^*=5$	7064.58
$k = 1500$	3	$t_0^*=0$, $s_1^*=1.666$, $t_1^*=1.666$, $s_2^*=3.333$, $t_2^*=3.333$, $s_3^*=5$, $t_3^*=5$	7814.58
$c = 2.5$	3	$t_0^*=0$, $s_1^*=1.25$, $t_1^*=1.666$, $s_2^*=2.916$, $t_2^*=3.333$, $s_3^*=4.583$, $t_3^*=5$	6266.41
$c = 3.75$	3	$t_0^*=0$, $s_1^*=1.666$, $t_1^*=1.666$, $s_2^*=3.333$, $t_2^*=3.333$, $s_3^*=5$, $t_3^*=5$	6314.58
$c = 6.25$	3	$t_0^*=0$, $s_1^*=1.666$, $t_1^*=1.666$, $s_2^*=3.333$, $t_2^*=3.333$, $s_3^*=5$, $t_3^*=5$	6314.58
$c = 7.5$	3	$t_0^*=0$, $s_1^*=1.666$, $t_1^*=1.666$, $s_2^*=3.333$, $t_2^*=3.333$, $s_3^*=5$, $t_3^*=5$	6314.58
$c_1 = 1.5$	2	$t_0^*=0$, $s_1^*=1.428$, $t_1^*=2.5$, $s_2^*=3.928$, $t_2^*=5$	5559.6
$c_1 = 2.25$	3	$t_0^*=0$, $s_1^*=1.666$, $t_1^*=1.666$, $s_2^*=3.333$, $t_2^*=3.333$, $s_3^*=5$, $t_3^*=5$	5967.71
$c_1 = 3.75$	3	$t_0^*=0$, $s_1^*=1.666$, $t_1^*=1.666$, $s_2^*=3.333$, $t_2^*=3.333$, $s_3^*=5$, $t_3^*=5$	6661.46
$c_1 = 4.5$	4	$t_0^*=0$, $s_1^*=1.25$, $t_1^*=1.25$, $s_2^*=2.5$, $t_2^*=2.5$, $s_3^*=3.75$, $t_3^*=3.75$, $s_4^*=5$, $t_4^*=5$	7006.25
$c_2 = 2.5$	3	$t_0^*=0$, $s_1^*=1.666$, $t_1^*=1.666$, $s_2^*=3.333$, $t_2^*=3.333$, $s_3^*=5$, $t_3^*=5$	5351.04
$c_2 = 3.75$	3	$t_0^*=0$, $s_1^*=1.666$, $t_1^*=1.666$, $s_2^*=3.333$, $t_2^*=3.333$, $s_3^*=5$, $t_3^*=5$	5832.81
$c_2 = 6.25$	3	$t_0^*=0$, $s_1^*=1.538$, $t_1^*=1.666$, $s_2^*=3.205$, $t_2^*=3.333$, $s_3^*=4.871$, $t_3^*=5$	6788.94
$c_2 = 7.5$	3	$t_0^*=0$, $s_1^*=1.111$, $t_1^*=1.666$, $s_2^*=2.777$, $t_2^*=3.333$, $s_3^*=4.444$, $t_3^*=5$	7085.42

5. Summary and conclusions

This paper proposes a finite-planning horizon extension of the basic two-product EOQ inventory model of Drezner et al. [11]. The demand of the two products were assumed to be changing with time. Inventory policies which determine the timing as well as the frequency of orders were considered. When the demand rates of the two products are proportional to each other and logconcave, a unique optimal replenishment schedule is shown to exist. For a fixed number of orders, the timings of the orders is the stationary point of the total inventory cost function. Numerical examples were also presented. Moreover, the possibility of relaxing a number of technical assumptions of the model were also discussed. Of particular interest is the inclusion of some penalty cost if the option of substituting a product is taken up. This may result in the selection of a policy of full substitution in some replenishment cycles (phenomenon which was absent in the basic model in [11]). Also, this inclusion lead to a dramatic change in the optimization problem. This problem is challenging due to the non-smoothness of the objective function.

In practice, the demand for products depends largely on time. The mathematical formulation and techniques presented in this work consider arbitrary functions of time for the substitutable products, which allows the decision maker to evaluate the consequences of a varied range of policies by employing only one model. Also, since the occurrence of temporary stock-outs at retail is common for various products, the proposed model reflects a number of practical concerns with regards to product substitution and so may assist retailers to handle more efficiently the inventory of these products.

In this paper we assumed that the replenishment rate is infinite. However in many cases inventory replenishment is restricted by a finite production rate. When a number of products are produced by the same machine with finite rate, direct joint replenishment is not generally feasible. In such conditions the presence of substitution between products may mitigate the shortage effects. Also an interesting extension to the proposed model could be a model where the two products are purchased from different suppliers (are not ordered jointly in each replenishment cycle) and in addition sup-plier selection issues could be considered. Also possible extensions of the proposed model may include:

1. models with deteriorating items: see Bakker et al. [44],
2. models with stock dependent items: see Urban [45],
3. models with inflation or permissible delay in payment as in Gilding [46] and [37].

It seems that search for the optimal policies in extensions (1) and (2) above will lead to optimization problems similar to the ones treated in this paper. However, extension (3) may not be straightforward.

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A. Proof of Theorem 1

Assume that the optimal solution in cycle i is given by $(t_{i-1}^*, \tilde{s}_i, t_i^*)$, where $t_{i-1}^* \leq \tilde{s}_i < t_i^*$, and $\tilde{s}_i \neq s_i^*$. Define the function

$$
\begin{aligned}
J(s) := &\int_{t_{i-1}^*}^{t_i^*} \{c_1 D_1(t) + c_2 D_2(t)\}(t - t_{i-1}^*)dt \\
&+ \int_s^{t_i^*} \{c - (c_2 - c_1)(t - t_{i-1}^*)\} D_2(t)dt.
\end{aligned}
$$

It follows that

$$
J'(s) = -\{c - (c_2 - c_1)(s - t_{i-1}^*)\} D_2(s).
$$

It is clear that $J'(s) \leq 0$, if $\gamma \leq 0$, or if $\gamma \geq t_i^* - t_{i-1}^*$ with unique minimum for $J(s)$ at $s = t_i^*$. Also, the minimum is unique if $t_{i-1}^* < t_{i-1}^* + \gamma \leq t_i^*$ and occurs at $s = t_{i-1}^* + \gamma$. Therefore, the function V has a unique minimum in $[t_{i-1}^*, t_i^*]$ given by s^*: see (15), in other words $V(t_{i-1}^*, s_i^*, t_i^*) < V(t_{i-1}^*, \tilde{s}_i, t_i^*)$. This leads to a contradiction with the the assertion that $(t_{i-1}^*, \tilde{s}_i, t_i^*)$ is optimal.

B. Proof of Lemma 1

The functions R_1 is continuously differentiable in $\bar{\Omega}$. Also, R_2 is differentiable in the space $\{(x, y) : x < y \leq H, \text{ and } x + \gamma < y\}$, and the space $\{(x, y) : x < y \leq H, \text{ and } x + \gamma > y\}$. Therefore it suffices to check that R_2 is differentiable on the line segment $(x, x + \gamma)$. Computation, using (19), shows that

$$
\begin{aligned}
\partial_x r_2(x + \gamma, y) &= -\int_{x+\gamma}^{y} D_2(t)dt, \\
\partial_y r_2(x + \gamma, y) &= \{y - (x + \gamma)\} D_2(y).
\end{aligned}
$$

Clearly $\partial_x r_2(x+\gamma, x+\gamma) = 0$, and $\partial_y r_2(x+\gamma, x+\gamma) = 0$. This leads, by (21) to the required result.

C. Proof of Lemma 2

We shall only show the lemma for $j = 2$. The proof for $j = 1$ is similar. Recall (36), (37), (19) and (24)-(25) to get

$$\begin{aligned}\mathcal{L}_x^{(2)} r_2(x,y) &= D_2(x) - D_2(y) + \frac{f_2'(x)}{f_2(x)} \int_x^y D_2(t)dt \\ &= \int_x^y \{f_2(t) - f_2(x)\} D_2(t)dt,\end{aligned} \tag{C1}$$

and

$$\mathcal{L}_x^{(2)} r_2(x+\gamma, y) = \int_{x+\gamma}^y \{f_2(t) - f_2(x)\} D_2(t)dt. \tag{C2}$$

Also, direct algebra shows that $\mathcal{L}_y^{(2)} r_2(x,y) = 0$ and the assumption that f_2 is non-decreasing. The proof is then immediate by using the definition of R_2 in (21).

D. Proof of Theorem 2

The proof is similar in spirit to that found in [37]. We shall only include details up to some key step related to Assumption A1.

The proof is based on the following induction hypothesis:

The Induction Hypothesis:

There exist functions $\{\tau_j\}, j = 0, \cdots, n-1$ which are piecewise continuously differentiable such that $\tau_0 \equiv 0$,

$$\tau_j(0) = 0, \tag{D1}$$

and

$$0 < \tau_j'(\eta^{\pm}) < 1 \quad \textit{for } 0 < \eta \le H \tag{D2}$$

and $j = 1, 2, \ldots, n-1$ *with the following properties. For every* $0 < h \le H$, *the minimum of* G_n *under the constraint*

$$0 = t_0 < t_1 < \cdots < t_n = h \tag{D3}$$

exists, is unique, and is given by

$$t_j = \tau_j(t_{j+1}) \quad \textit{for } j = n-1, n-2, \ldots, 1, 0.$$

Furthermore, the minimum value $g_n(h)$ *satisfies*

$$g_n'(h) = (\partial_y R)(\tau_{n-1}(h), h). \tag{D4}$$

End of The Induction Hypothesis

Let $n = 1$, then (42) and (23) give $G_n(t_0, t_1) = R(0, h)$, and $g_n(h) = R(0, h)$. It is clear that the theorem is true in this case. Assume now that the theorem is true for $n \ge 1$, and consider the problem of minimizing $G_{n+1}(t_0, \cdots, t_{n+1})$ with $t_0 = 0 < t_1 < \cdots < t_{n+1} = h \le H$.
The Dynamic Programming principle of Bellman leads to:

$$\sigma_{n+1}(\eta, h) = g_n(\eta) + R(\eta, h). \tag{D5}$$

We aim to minimize σ_{n+1} under the constraint $0 < \eta < h$. This function is continuous on the interval $[0, h]$. Also,

$$\begin{aligned}(\partial_x \sigma_{n+1})(\eta, h) &= g_n'(\eta) + (\partial_x R)(\eta, h) \\ &= (\partial_y R)(\tau_{n-1}(\eta), \eta) + (\partial_x R)(\eta, h).\end{aligned} \tag{D6}$$

This is justified by (D4) in the induction hypothesis.

Now, again by (D1) in the induction hypothesis and the definition of R

$$\begin{aligned}(\partial_x\sigma_{n+1})(0,h) &= (\partial_y R)(\tau_{n-1}(0),0) + (\partial_x R)(0,h) \\ &= (\partial_y R)(0,0) + (\partial_x R)(0,h) \\ &= \sum_{j=1}^{2}\{(\partial_y R_j)(0,0) + (\partial_x R_j)(0,h)\}.\end{aligned}$$

Using the definitions of R_1 and R_2 in (20), and (21) respectively together with (24), and (25) we assert that $(\partial_x\sigma_{n+1})(0,0) < 0$. Likewise, we get

$$\begin{aligned}(\partial_x\sigma_{n+1})(h,h) &= (\partial_y R)(\tau_{n-1}(h),h) + (\partial_x R)(h,h) \\ &= \sum_{j=1}^{2}\{(\partial_y R_j)(\tau_{n-1}(h),h) + (\partial_x R_j)(h,h)\} > 0.\end{aligned}$$

It follows from the continuity that there exist at least one $\eta \in (0,h)$ such that

$$(\partial_x\sigma_{n+1})(\eta,h) = 0. \tag{D7}$$

Note by Remark 3 that may differentiate $(\partial_x\sigma_{n+1})(\eta,h)$ with respect to η and h except at the points $\tau'_{n-1}(\eta^+) \neq \tau'_{n-1}(\eta^-)$, $\tau_{n-1}(\eta)+\gamma = \eta$, and $\eta+\gamma = h$. Denote such points η, where $\tau'_{n-1}(\eta^+) \neq \tau'_{n-1}(\eta^-)$, by $\xi_k, k = 1, \cdots, m$. Also, note that by the induction hypothesis there exists at most one point where $\tau_{n-1}(\eta)+\gamma = \eta$, denote this point by ξ_0. Excluding all points $\eta = \xi_k, k = 0, \cdots, m$, and $\eta+\gamma = h$, we differentiate $(\partial_x\sigma_{n+1})(\eta,h)$ with respect to η to get

$$\begin{aligned}(\partial_x^2\sigma_{n+1})(\eta,h) &= (\partial_x\partial_y R)(\tau_{n-1}(\eta),\eta)\tau'_{n-1}(\eta) \\ &\quad + (\partial_y^2 R)(\tau_{n-1}(\eta),\eta) + (\partial_x^2 R)(\eta,h).\end{aligned} \tag{D8}$$

Using the definition of R, ($\mathcal{L}_x^{(j)}$, $\mathcal{L}_y^{(j)}$, $j = 1,2$), (39), and (40), we may write (D8) as

$$(\partial_x^2\sigma_{n+1})(\eta,h) = -\{1-\tau'_{n-1}(\eta)\}(\partial_x\partial_y R)(\tau_{n-1}(\eta),\eta)$$

$$+ \{(\mathcal{L}_x R)(\eta, h) + (\mathcal{L}_y R)(\tau_{n-1}(\eta), \eta)\}$$
$$- f(\eta)(\partial_x \sigma_{n+1})(\eta, h) - (\partial_x \partial_y R)(\eta, h). \tag{D9}$$

Under assumptions (A1) and (A2) and invoking the induction hypothe-sis, Lemma 2, and (D7)

$$(\partial_x{}^2 \sigma_{n+1})(\eta, h) \geq -(\partial_x \partial_y R)(\eta, h) > 0. \tag{D10}$$

Also, letting $\eta \to \xi^{\pm}_{k}$, and $\eta = h - \gamma$, separately, we have

$$(\partial_x{}^2 \sigma_{n+1})(\eta^{\pm}, h) \geq -(\partial_x \partial_y R)(\eta^{\pm}, h) > 0. \tag{D11}$$

Therefore, (D7) has a unique solution. This solution is the global minimum of $\sigma_{n+1}(., h)$ for every $h \in (0, H]$.

The rest of the proof follows the steps as in [37] and is omitted.