# Inventory models for perishable items with advanced payment, linearly time-dependent holding cost and demand dependent on advertisement and selling price


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## Abstract

Advertisement is one of the most effective ways to spread out the popularity of the product for all categories of customers. Consequently, this has a direct impact to aggrandize product's demand to a great



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extent. On the other hand, if the maximum lifetime of a product is expired once then it can neither be usable nor be re-usable. Thus the date of maximum life time of the product is an essential issue in inventory management. Advance payment is another important factor among interrelation of suppliers and retailers for a highly demanding seasonal product. Combining these issues, two different inventory models for perishable items are formulated under linearly time-dependent increasing holding cost whereas demand of the product is dependent on the selling price of the product and the frequency of advertisement as well. In the first model, shortages are not considered whereas in the second one, partial backlogged shortages are incorporated. In both cases, the optimality of the proposed inventory models is discussed theoretically along with its solution algorithm. To validate the proposed models, three numerical examples are solved. Finally, the effect of changes of different parameters is studied numerically to perform a sensitivity analysis and a fruitful conclusion is done.

**Keywords:** Inventory; advertising; maximum lifetime; advance payment; partial backlogging

## 1. Introduction

Advertising of a product plays an important role in inventory management to create the brand awareness and to appraise the brand availability for the customers in different markets. In the advertisement, manufacturers/suppliers generally provide the information about their products especially, introduction of new product or modified product from the older one. With this information, customers are aware of the product and its use. So, the demand of any product is directly dependent on the impact of the advertisement. In this connection, manufacturers/suppliers want to publish the advertisements in a popular media such as print media, electronic media, among other ways with the help of modern technologies or popular persons in order to attract more and more people for purchasing the products. In this study, this factor is considered in the demand of the product.

It is well-known that every product has a maximum lifetime. Within this lifetime of the product, the product is in a useable condition for the customers but after the lifetime of the product this must not be consumable or used. Thus, it indicates that the expired product is either scrap or thrown in dustbins. If the products are not sold within the lifetime, then manufacturers/suppliers lose their all investments in these products. In this perspective, lifetime of the product has also huge impact on inventory management. In this connection, one can consider several perishable products such as all sorts of vegetables, cookies, fruits, cooked-foods, different ingredients for cooking items, sea-foods, meats, poultry, fishes, egg and all kinds of dairy products. All kinds of the above mentioned products lose their useable conditions totally after a certain time period. Consequently, these products are not in a useable condition after their expired date. With this reference, it should be noticed that the different products, like electronic gadget, gasoline, alcohol, medicine, dried food, rice, wheat, among others, which deteriorate or decline over time have also an expiry period after which they lose their usability and hence these are not in a condition for sale. Very few research works have been done by considering this concept. In the proposed work, this factor is included in the formulation of the inventory model.

For the highly demandable products, the manufacturers or suppliers sometimes request that retailer pays in advance a portion of the total purchase cost. In this advance payment scheme, both parties are benefited in different ways. In this way, the suppliers or manufacturers reduce the risk of cancelling the orders by receiving prepayments. On the other hand, retailer pays the advance payment with alacrity in order to ensure on-time delivery. In response to the advance payment, suppliers or manufacturers offer different price discounts or some credit facilities or some other kinds of facilities in order to promote their business.

In most of the traditional EOQ inventory models, the carrying cost per unit time is considered as constant over the entire replenishment cycle. However, in practice, this assumption is not suitable for all perishable items. Since the deterioration rate increases as much as the product approaches to the maximum lifetime or expired date, the carrying cost per unit item increases continuously to preserve the freshness or the usable condition of the products. To the best of our knowledge, no work has been reported in the literature considering this kind of time-varying carrying cost for the maximum lifetime related perishable items under advance payment policy.

### *1.1 Literature review*

The manufacturers/suppliers/retailers provide different kinds of offers or publish the advertisement of the product in order to attract more customers. So, they use the popular media such as social media, television, newspaper, cinema, poster, etc. Also, the selling price of an item is one of the decisive factors in selecting an item for purchasing. For the first time, this concept of a business technique for decision making was introduced by Kotler (1971). He introduced the relationship between pricing and EOQ inventory model. After that, Ladany and Sternleib (1974) investigated an EOQ inventory model with the variation of selling price effect. Lately, Subramanyam and Kumaraswamy (1981), Urban (1992), Goyal and Gunasekaran (1995), Bhunia and Shaikh (2011), Shah et al. (2013), Bhunia and Shaikh (2014), Shaikh et al. (2017), Panda et al. (2019) and others introduced some inventory models by considering the effect of advertisement on demand.

Advance payment is another important issue in inventory management. Due to competitive market situations and uncertainty of customers, it is observed that supplier requires some advance payment from his retailer. The retailer pays the advance payment with the aim of having a guarantee of an on-time delivery. In response to the advance payment, supplier provides price discount or credit facility or other kind of facility in order to stimulate the business. To the best of our knowledge, this type of concept was introduced by Gupta et al. (2009). Inventory model with stochastic lead time and price dependent demand under advance payment has been introduced by Maiti et al. (2009). In this area, after the work of Maiti et al. (2009), one may refer to the works of Guria et al. (2013), Taleizadeh et al. (2013), Taleizadeh(2014a), Teng et al. (2016), Lashgari et al. (2016), Wu et al. (2016), Taleizadeh (2017), Tavakoli and Taleizadeh (2017), Khan et al. (2019a), Khan et al. (2020), among others.

Every product has a certain lifetime. During this fixed lifetime, the customers can use this specific product for their requirement. After this fixed lifetime, the product is totally unusable and it has lost its value in the markets. Here, some expiration related works are mentioned. Hsu et al. (2006) first introduced expiration concept in an inventory model. After that, Hsu et al. (2007) modified their inventory model by using uncertain lead time. Jain and Singh (2011) extended the inventory model of Hsu et al. (2007) using several concepts such as multi-echelon, inflation, expiration date dependent deterioration, etc. Wu et al. (2014) introduced two-level trade credit in an expiration system. Chen and Teng (2015) proposed another new concept such as upstream and downstream delay in the payment. Wu et al. (2016) considered the freshness concept of the product in the inventory system. Teng et al. (2016) presented the advance payment concept for the expiration date product related to inventory system. Feng et al. (2017) proposed expiration related inventory problem like Wu et al. (2016) except freshness of the product.

Most of the deteriorating items do not have constant deterioration rate; in fact the deterioration continuously increases with time. For these items, consequently, the carrying cost per unit item may not be constant in the whole storage time and this must be an increasing function of the storage time. In this line, Ferguson et al. (2007) proposed an inventory model with non-linear carrying cost per unit item based on the storage time. Alfares (2007) studied an inventory model with an increasing holding cost for the stock-level dependent demand rate. Assuming both the demand rate and carrying cost per unit dependent on current stock level, Pando et al. (2012) developed an EOQ inventory model. After that, Pando et al. (2013) extended their research work to a profit maximization inventory model by considering that the carrying cost is modeled as a non-linear function of the current stock level and the duration of the storage time as well. Recently, Shaikh et al. (2019) described a mathematical model for a deteriorating product under all-units discount environment considering the per unit holding cost as a linearly increasing function of the storage period.

Selling price of the product is another important factor in inventory management. When a product is launched into the market manufacturer is very much alert about the selling price of the product. If the price of the product is very high, then the common people are not able to buy this product. So, its impact is directly reflected to the demand of the product. In this connection, one may refer to the recent works of Sana (2011), Avinadav et al. (2013), Bhunia and Shaikh (2014), Ghorieshi et al. (2015), Alfares and Ghaithan (2016), Tiwari et al. (2018), Khan et al. (2019b), among others. Finally, some of the related works are presented in Table 1.

**Table 1.**Recent research works in the inventory literature.

| **Authors** | **Demand** | **Shortage** | **Payment** | **Deterioration** | **Advertisement of the product** | **Holding cost** |
|---|---|---|---|---|---|---|
| Thangam (2012) | Constant | No | Advanced and delayed payment | Constant | No | Constant |
| Guira et al. (2013) | Variable demand | Full | Advanced payment | Constant | No | Constant |
| Taleizadeh et al. (2013a) | Constant | Partial | Delayed payment | No | No | Constant |
| Taleizadeh. et al. (2013b) | Constant | Full | No | Constant | Yes | Constant |
| Taleizadeh (2014a) | Constant | Partial | Advanced Payment | Constant | No | Constant |
| Taleizadeh (2014b) | Constant | No shortage, Full | Advanced Payment | Constant | No | Constant |
| Taleizadeh et al. (2015) | Constant and price sensitive | No | No | No | No | Constant |
| Chen and Teng (2015) | Credit period dependent | No | Delayed payment | Expiration | No | Constant |
| Teng et al. (2016) | Constant | No | Advanced Payment | Time-varying | No | Constant |
| Lashgari et al. (2016) | Constant | No shortage, Full, Partial | Advanced and delayed payment | No | No | Constant |
| Wu et al. (2016) | Constant | Partial | Advanced Payment | Time-varying | No | Constant |
| Alfares and Ghaithan (2016) | Price dependent | No | No | No | No | Linearly time varying |
| Feng et al. (2017) | Price, freshness and inventory level | No | No | Expiration | No | Constant |
| Taleizadeh (2017) | Constant | Partial | Advanced Payment | No | No | Constant |
| Tavakoli and Taleizadeh (2017) | Constant | Full | Advanced | No | No | Constant |
| Pervin et al. (2017) | Stock dependent | Partial | Delayed payment | Constant | No | Linearly time varying |

| | | | | | | |
|---|---|---|---|---|---|---|
| Pervin et al. (2018a) | Stock dependent | Partial | Delayed payment | Constant | No | Linearly time varying |
| Mashud et al. (2018) | Price and stock-dependent | Partial | No | Constant | No | Constant |
| Pervin et al. (2018b) | Time-dependent | Partial | No | Constant | No | Linearly time varying |
| Tiwari et al. (2018) | Price dependent demand; | Partial | Delayed payment | Expiration | No | Constant |
| Pervin et al. (2019) | Price- and stock-dependent | Partial | Delayed payment | Constant | No | Constant |
| Khan et al. (2019a) | Price dependent | Partial | Advanced Payment | Constant | No | Constant |
| This paper | Advertisement and price dependent demand; where price is a decision variable | Zero ending and partial backlogging | Advanced Payment | Expiration | Yes | Linearly time varying |

In this work, two inventory models (with and without shortages) are formulated for perishable items which deteriorate continuously with maximum life time. In addition, the following factors are also included:

(i) The demand is dependent on the frequency of advertisement and selling price of the product.

(ii) The carrying cost follows a linearly time-dependent increasing function.

(iii) In shortages case, partial backlogging is considered with variable rate dependent on the length of waiting time of the customer.

(iv) There is an advanced payment policy.

The corresponding optimal solution, for both inventory models, is proved theoretically. In order to solve each inventory model, a solution algorithm is developed. With the aim to illustrate and also to validate the inventory models, three numerical examples are solved with the help of proposed algorithms. Finally, a sensitivity analysis is carried out with respect to different system parameters for the shortage inventory model.

## 2. Notation and Assumptions

To describe our system, the following notation and assumptions are presented.

*2.1 Notation*

| Symbol | Description |
|---|---|

| | |
|---|---|
| $p$ | selling price per unit ($/unit) |
| $C_0$ | replenishment cost per order ($/order) |
| $C_p$ | purchasing cost per unit ($/unit) |
| $C_s$ | shortage cost per unit ($/unit) |
| $C_l$ | opportunity cost per unit ($/unit) |
| $G$ | cost per advertisement ($/advertisement) |
| $g$ | constant part of the unit holding cost ($/unit/time unit) |
| $h$ | coefficient of linearly time dependent cost to hold a unit ($/unit/(time)$^2$) |
| $a$ | location parameter of demand rate $(a > 0)$ |
| $b$ | shape parameter of demand rate $(b > 0)$ |
| $\theta(t)$ | time varying deterioration rate at any instant $t$ $(0 < \theta(t) << 1)$ (/unit/time unit) |
| $L$ | length of the lead time during which prepayments are paid (time unit) |
| $E$ | the maximum lifetime or expired date (time unit) |
| $I_c$ | interest rate of the cost of loan (% / time unit) |
| $n$ | number of equally spaced prepayments during the lead time |
| $\alpha$ | fraction of the purchasing cost to be paid with multiple prepayments $(0 < \alpha < 1)$ |
| $\gamma$ | advertising elasticity of the demand function |
| $B(t)$ | partial backlogging rate which is a function of waiting time $t$ |
| $\delta$ | backlogging parameter ( $\delta > 0$ ) |
| $S$ | the initial inventory level (units) |
| $R$ | maximum amount of shortage level for second model |
| $Q$ | order quantity per replenishment cycle (units) |
| $t_1$ | stock-in period for the inventory model with shortages (time unit) |
| $t_2$ | time period of shortages (time unit) |
| $T$ | cycle length (for shortages case $T = t_1 + t_2$ ) (time unit) |
| $q(t)$ | inventory level at any time $t$ (units) |
| $\Pi_1(A, p, T)$ | the total profit for the inventory model without shortages ($/time unit) |
| $\Pi_2(A, p, t_1, t_2)$ | the total profit for the inventory model with partial backlogging ($/time unit) |
| ***Decision variables*** | |
| $A$ | frequency of advertisement |
| $p$ | selling price per unit ($/unit) |
| $t_1$ | stock-in period for the inventory model with shortages (time unit) |
| $t_2$ | time period of shortages (time unit) |
| $T$ | cycle length for the inventory model without shortages (time unit) |

### *2.2 Assumptions*

1. A single deteriorating item is considered.
2. Replenishment rate is infinite.
3. Customers always try to purchase the products with affordable prices. Obviously, a higher selling price attenuates the customers' demand in a great extent. Consequently, the demand for any product can be modeled as a linearly decreasing function of the selling price. Moreover, the advertisement of any product is one of the best ways not only to get acquainted with its customers but also to spread out the popularity of the product to all categories of customers. Due to advertisement, demand will increase and product will be sold-out quickly. That would be ideal in the case of perishable items, where their life span is short. In this perspective, it is considered advertisement elasticity rather than price elasticity. The advertisement of a product is directly proportional to the demand. In this reason, we have considered the advertisement elasticity in order the demand increases quickly. Consequently, the demand of the product can be considered as a multiplicative form in the following way: $D(A,p)=(A+1)^{\gamma}(a-bp)$,

   where $A\in\{0,1,2,3,.....\}$ is the frequency of advertisement, $\gamma\in[0,1)$ is the advertising elasticity of the demand function, $a$ is the location parameter of demand rate, $b$ is the shape parameter of the demand and $D(A,p)\geq 0$.

   It is noteworthy that, the frequency of advertisement ($A$) is either zero or a positive integral number.
4. The cost of holding a unit of the item in the warehouse is an increasing function of the storage time. The holding cost consists of two parts, namely, constant part to hold a unit and linearly time-dependent part to hold a unit, i.e., $H(t)=g+ht$ where $g,h>0$.
5. The deterioration rate $\theta(t)$ is a continuously increasing function of holding time $t$ of the product. This rate increases as much as the product approaches to its maximum lifetime or expired date $E$; and finally it reaches to 100% at time $t$=$E$. The deterioration rate function $\theta(t)$ is expressed as follows $\theta(t)=\dfrac{1}{1+E-t}$, $0\leq t\leq T\leq E$ as in Sarker (2012) and Teng et al. (2016). Remarkable point is that the upper limit of the replenishment cycle length $T$ is the maximum life time $E$, which reveals that after the maximum lifetime product cannot be sold.
6. Any sold or deteriorated product cannot be returned or repaired.
7. Shortages are allowed with a partial backlogging rate $B(y)$, a decreasing function of the waiting time $y$.That is $B(y)=1/1+\delta y$ and $0\leq B(y)\leq 1$with $B(0)=1$, here $y$ indicates the waiting time up to the arrival of next lot and $\delta>0$. If $\delta=0$ then the inventory model with partial backlogging reduces to a fully backlogging inventory model.

8. The supplier requests his retailer to make the order by paying a certain $\alpha$ fraction of the total purchasing cost by $n$ equal multiple installments at equal intervals in $L$ units of time prior to the delivery time and receive the lot by paying the remaining portion (1-$\alpha$) of the total purchasing cost at the receiving time.

## 3. Mathematical formulation of the inventory models

In this section the inventory models, with and without shortages, are developed.

*3.1 The inventory model without shortages*

The retailer makes an order of $Q$ units by paying a certain fraction ($\alpha$) of the total purchasing cost by $n$ equal installments at equal intervals in $L$ units of time prior to the delivery time from the supplier. The retailer receives the lot at time $t$=0 by paying the remaining portion (1-$\alpha$) of the total purchasing cost. Thereafter, the lot is ready to be consumed in order to fulfill customers' demand. The inventory level gradually decreases to zero over the period [0, $T$] due to the combined effect of customers' demand and deterioration. The inventory system is depicted in Figure 1.

The inventory level at any instant $t$ during the time interval [0, $T$] is governed by the following differential equation:

$$\frac{dq(t)}{dt}+\frac{1}{1+E-t}q(t)=-(A+1)^{\gamma}(a-bp)\,, \tag{1}$$

with boundary conditions $q(0)=Q$ and $q(T)=0$.

Considering that $q(T)=0$ then the solution of Eq. (1) is given by:

$$q(t)=(A+1)^{\gamma}(a-bp)(1+E-t)\ln\left|\frac{1+E-t}{1+E-T}\right|. \tag{2}$$

As $q(0)=Q$, the order quantity $Q$ per replenishment cycle is given by:

$$Q=(A+1)^{\gamma}(a-bp)(1+E)\ln\left|\frac{1+E}{1+E-T}\right|. \tag{3}$$

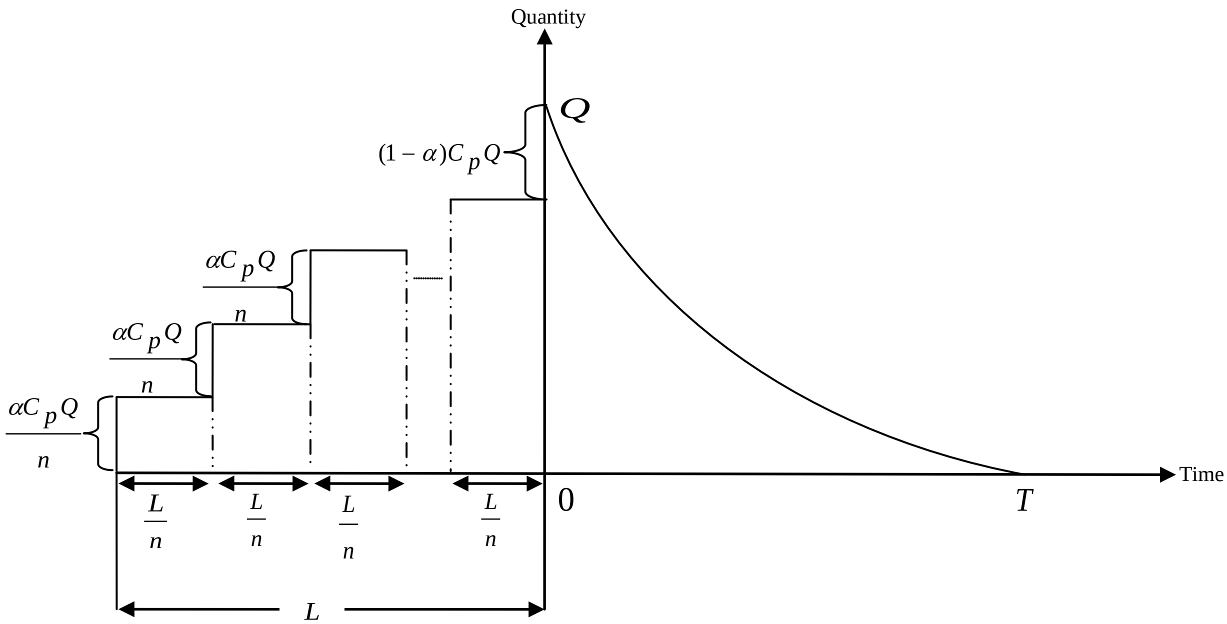


**Figure 1.** Graphical presentation of the inventory model without shortages.

The relevant components of the profit function are as follows:

i) Ordering cost ($OC$): $C_0$.

ii) Advertisement Cost ($AC$): $AG$.

iii) Purchasing cost ($PC$): $C_p\left[(A+1)^{\gamma}(a-bp)(1+E)\ln\left|\frac{1+E}{1+E-T}\right|\right]$.

iv) Interest of loan ($IL$): The interest of loan from Figure 1 is given by:

$$IL = I_c\left[\left(\frac{\alpha C_p Q}{n}\right)\frac{L}{n}(1+2+3+...+n)\right] = \left(\frac{n+1}{2n}\right)I_c\alpha L C_p\left[(A+1)^{\gamma}(a-bp)(1+E)\ln\left|\frac{1+E}{1+E-T}\right|\right].$$

v) Holding cost ($HC$):

$$\int_0^T (g+ht)q(t)dt = g\int_0^T q(t)dt + h\int_0^T tq(t)dt$$

$$=\left[\frac{g}{2}(1+E)+\frac{h}{6}(1+E)^2\right]Q + g(A+1)^{\gamma}(a-bp)\left[\frac{T^2}{4}-\frac{(1+E)T}{2}\right] + h(A+1)^{\gamma}(a-bp)\left[\frac{T^3}{9}-\frac{(1+E)T^2}{12}-\frac{(1+E)^2T}{6}\right].$$

vi) Sales revenue ($SR$): $p\int_0^T (A+1)^{\gamma}(a-bp)dt = p(A+1)^{\gamma}(a-bp)T$.

Thus, the profit function per unit time is given by:

$$\Pi_1(A,p,T) = \frac{1}{T}\left[SR - OC - AC - PC - IL - HC\right]$$

$$=\frac{1}{T}\left[(A+1)^{\gamma}(a-bp)XT-\left(C_0+GA+YQ+g(A+1)^{\gamma}(a-bp)\frac{T^2}{4}+h(A+1)^{\gamma}(a-bp)\left\{\frac{T^3}{9}-\frac{(1+E)T^2}{12}\right\}\right)\right], (4)$$

where $X=p+\frac{(1+E)g}{2}+\frac{(1+E)^2h}{6}$ and $Y=\left(1+\frac{n+1}{2n}I_c\alpha L\right)C_p+\frac{g}{2}(1+E)+\frac{h}{6}(1+E)^2$.

To ensure $D(A,p)\geq 0$, the selling price per unit $p$ must be less than or equal to $\frac{a}{b}$. In this case, the main objective is to maximize the retailer's total profit per unit time $\Pi_1(A,p,T)$ with respect to the decision variables: frequency of advertisements (*A*), selling price per unit (*p*) and replenishment cycle length (*T*) under the constraint $p-\frac{a}{b}\leq 0$. Therefore, the problem reduces to the following mixed integer non linear optimization problem:

Maximizing $\Pi_1(A,p,T)$

subject to $0<p-\frac{a}{b}\leq 0, 0<T\leq E$ and $A(\geq 0)$ is an integer.

*3.2 The inventory model with shortages*

Here, an order of *Q*=(*S*+*R*) units is made by the retailer. He pays a certain *α* fraction of the total purchasing cost by $n$ equal installments at equal intervals in *L* units of time before to the delivery time from his/her supplier. At time *t*=0, the retailer's on-hand inventory level becomes *S* after fulfilling the total accumulated backlogged shortages *R* immediately. Due to the resultant effect of both the customers' demand and deterioration, the stock level decreases continuously, and finally, drops at zero level at time $t=t_1$. Then, the shortages are accumulated depending on the waiting time of the customers during the time interval $[t_1,t_1+t_2]$. Figure 2 depicts the behavior of the inventory system over the entire cycle length.

Now, the inventory system is modeled with the following governing differential equations:

$$\frac{dq(t)}{dt}+\frac{1}{1+E-t}q(t)=-(A+1)^{\gamma}(a-bp), 0<t\leq t_1, \qquad (5)$$

and

$$\frac{dq(t)}{dt}=-\frac{(A+1)^{\gamma}(a-bp)}{1+\delta(t_1+t_2-t)}, t_1<t\leq t_1+t_2, \qquad (6)$$

with the conditions $q(0)=S$, $q(t_1)=0$ and $q(t_1+t_2)=-R$.

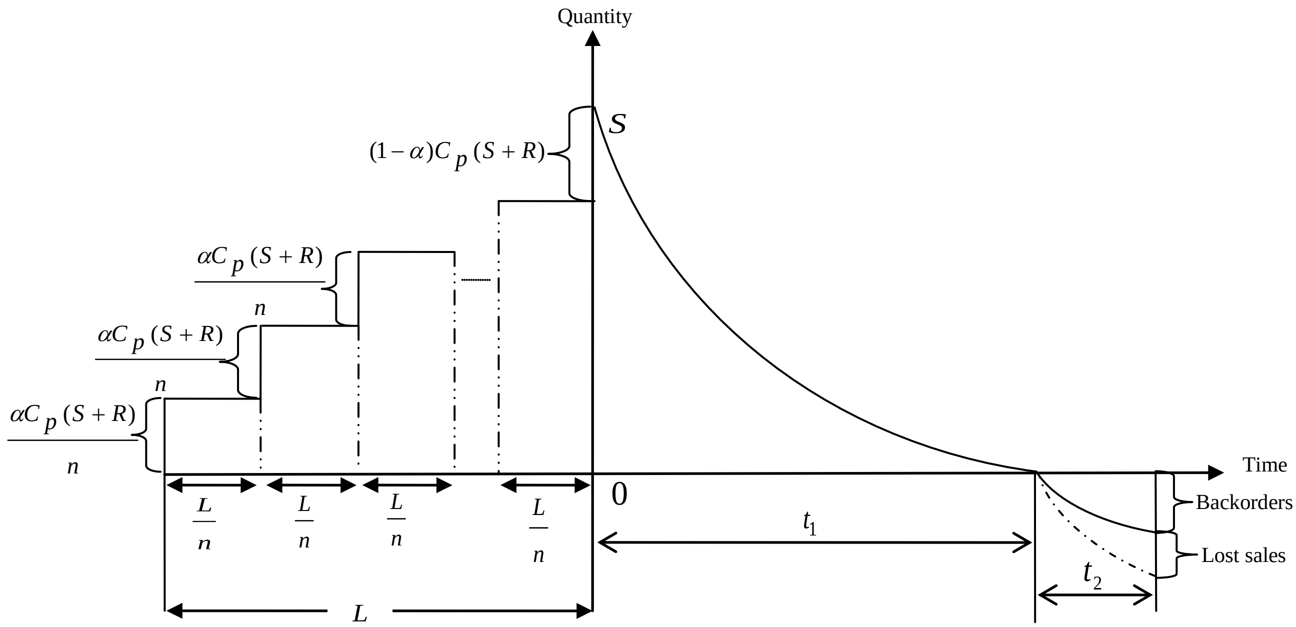


**Figure 2.** Graphical presentation of the inventory model with shortages.

The solutions of Eq. (5) and Eq. (6) are expressed as follows:

$$q(t)=(A+1)^{\gamma}(a-bp)(1+E-t)\ln\left|\frac{1+E-t}{1+E-t_1}\right|,\ 0<t\le t_1, \tag{7}$$

and

$$q(t)=\frac{(A+1)^{\gamma}(a-bp)}{\delta}\ln\left|1+\delta(t_1+t_2-t)\right|-R,\ t_1<t\le t_1+t_2. \tag{8}$$

Since $q(0)=S$ at $t=0$, hence

$$S=(A+1)^{\gamma}(a-bp)(1+E)\ln\left|\frac{1+E}{1+E-t_1}\right|. \tag{9}$$

The maximum amount of shortages, with the help of continuity of $q(t)$ at $t=t_1$, is given by:

$$R=\frac{1}{\delta}(A+1)^{\gamma}(a-bp)\ln\left|1+\delta t_2\right|. \tag{10}$$

Therefore, the lot size is given by:

$$Q=(A+1)^{\gamma}(a-bp)\left[(1+E)\ln\left|\frac{1+E}{1+E-t_1}\right|+\frac{1}{\delta}\ln\left|1+\delta t_2\right|\right]. \tag{11}$$

The relevant components of the total profit function are given by:

i) Ordering cost ($OC$): $C_0$.

ii) Advertisement cost ($AC$): $AG$.

iii) Purchasing cost ($PC$): $C_p(A+1)^\gamma(a-bp)\left[(1+E)\ln\left|\frac{1+E}{1+E-t_1}\right|+\frac{1}{\delta}\ln\left|1+\delta t_2\right|\right].$

iv) Interest of loan ($IL$): The interest of loan from Figure 2 is given by:

$$IL = I_c\left[\left(\frac{\alpha C_p Q}{n}\right)\frac{L}{n}(1+2+3+...+n)\right]=\left(\frac{n+1}{2n}\right)I_c\alpha LC_p(A+1)^\gamma(a-bp)\left[(1+E)\ln\left|\frac{1+E}{1+E-t_1}\right|+\frac{1}{\delta}\ln\left|1+\delta t_2\right|\right].$$

v) Holding cost ($HC$): $\int_0^{t_1}(g+ht)q(t)dt = g\int_0^{t_1}q(t)dt+h\int_0^{t_1}tq(t)dt$

$$HC=\left[\frac{g}{2}(1+E)+\frac{h}{6}(1+E)^2\right](A+1)^\gamma(a-bp)(1+E)\ln\left|\frac{1+E}{1+E-t_1}\right|$$
$$+g(A+1)^\gamma(a-bp)\left[\frac{t_1^2}{4}-\frac{(1+E)t_1}{2}\right]+h(A+1)^\gamma(a-bp)\left[\frac{t_1^3}{9}-\frac{(1+E)t_1^2}{12}-\frac{(1+E)^2t_1}{6}\right].$$

vi) Shortage cost ($SC$): $C_s\int_{t_1}^{t_1+t_2}\left[-q(t)\right]dt=\frac{C_s(A+1)^\gamma(a-bp)}{\delta}\left[t_2-\frac{\ln\left|1+\delta t_2\right|}{\delta}\right].$

vii) Lost sale cost ($LSC$): $C_l\int_{t_1}^{t_1+t_2}\left[1-\frac{1}{1+\delta t_2}\right](A+1)^\gamma(a-bp)dt=C_l(A+1)^\gamma(a-bp)\left[t_2-\frac{\ln\left|1+\delta t_2\right|}{\delta}\right].$

viii) Sales revenue ($SR$): $p\int_0^{t_1}(A+1)^\gamma(a-bp)dt+pR=(A+1)^\gamma(ap-bp^2)\left[t_1+\frac{1}{\delta}\ln\left|1+\delta t_2\right|\right].$

So, the total profit function per unit time is given by:

$$\Pi_2(A,p,t_1,t_2)=\frac{1}{t_1+t_2}\left[SR-OC-AC-PC-IL-HC-SC-LSC\right]$$
$$=\frac{1}{t_1+t_2}\left\{\begin{array}{l}(A+1)^\gamma(a-bp)Xt_1-C_0-GA-(A+1)^\gamma(a-bp)Y(1+E)\ln\left|\frac{1+E}{1+E-t_1}\right|-g(A+1)^\gamma(a-bp)\frac{t_1^2}{4}\\ -h(A+1)^\gamma(a-bp)\left[\frac{t_1^3}{9}-\frac{(1+E)t_1^2}{12}\right]+(A+1)^\gamma(a-bp)X_1\frac{\ln\left|1+\delta t_2\right|}{\delta}-\left(C_l+\frac{C_s}{\delta}\right)(A+1)^\gamma(a-bp)t_2\end{array}\right\}, \quad (12)$$

where $X=p+\frac{g(1+E)}{2}+\frac{h(1+E)^2}{6}$, $Y=\left(1+\frac{n+1}{2n}I_c\alpha L\right)C_p+\frac{g}{2}(1+E)+\frac{h}{6}(1+E)^2$ and

$X_1=p+C_l+\frac{C_s}{\delta}-\left(1+\frac{n+1}{2n}I_c\alpha L\right)C_p$.

In order to ascertain $a - bp \geq 0$, the selling price per unit $p$ must be less than or equal to $\frac{a}{b}$. Now, the objective is to find the optimal frequency of advertisements $A^*$, selling price $p^*$ per unit, the time $t_1^*$ at which the inventory level reaches zero and shortage period $t_2^*$ in order to maximize the retailer's total profit per unit time $\Pi_2(A, p, t_1, t_2)$ under the inequality constraint $p - \frac{a}{b} \leq 0$. Therefore, the problem becomes the following mixed integer non linear optimization problem:

Maximizing $\Pi_2(A, p, t_1, t_2)$

subject to $0 < p - \frac{a}{b} \leq 0, 0 < t_1 \leq E$ and $A(\geq 0)$ is an integer.

The next section discusses the optimality of the retailer's profit function and the derivation of the necessary and sufficient conditions for the inventory models.

## 4. Solution methodology

First of all, the optimality for the inventory model without shortage is derived and then that of for the inventory model with partial backlogging. In order to explore the concavity, the results of generalized concave functions in Cambini and Martein (2009, p.245) are used. According to these results, any fractional function of the form

$$\Psi(x) = \frac{f(x)}{g(x)}, x \in R^n \tag{13}$$

is (strictly) pseudo-concave, if $f(x)$ is concave and differentiable and $g(x)$ is positive and affine.

*4.1 The inventory model without shortages*

To prove the optimality theoretically, some theorems are stated and proved with the help of the Theorems 3.2.9 and 3.2.10 of Cambini and Martein (2009).These results state that for a given value of *p* and *A*, it can be shown that the profit function $\Pi_1(A, p, T)$ is a strictly pseudo-concave function of $T$. As a result, there is a unique optimal solution $T^*$ such that the total profit function per unit time $\Pi_1(A, p, T)$ is maximized.

In order to find the optimal replenishment cycle length $T^*$, setting the first order partial derivative of $\Pi_1(A, p, T)$ with respect to *T* equals to zero, we find:

$$C_0 + GA + (A+1)^\gamma (a - bp)\left[Y(1+m)\left\{\ln\left(\frac{1+E}{1+E-T}\right) - \frac{T}{(1+E-T)}\right\} - \frac{gT^2}{4} - hT^2\left\{\frac{2T}{9} - \frac{(1+E)}{12}\right\}\right] = 0. \tag{14}$$

To explore the existence of a unique value of $T \in (0, E]$ at which $\Pi_1(A, p, T)$ is maximized, for convenience, define:

$$\Delta_1 = (A+1)^{\gamma}(a-bp)\left[\frac{(C_0+GA)}{(\mathrm{A}+1)^{\gamma}(a-bp)E^2} + Y\frac{(1+E)}{E^2}\{\ln(1+E)-E\} - \frac{g}{4} - \frac{h}{36}(5E-3)\right]. \tag{15}$$

So, the following theorem is proposed.

**Theorem 1.** For any given positive values of $p$ and $A$;

(a) $\Pi_1(A, p, T)$ is a pseudo-concave function in $T$, and hence there exists a unique $T$ satisfying Eq. (14) such that $\Pi_1(A, p, T)$ is maximized.

(b) If $\Delta_1 > 0$, then the total profit per unit time $\Pi_1(A, p, T)$ attains its global maximum value at $T^* = E$.

(c) If $\Delta_1 \le 0$, then the total profit per unit time $\Pi_1(A, p, T)$ achieves its global maximum value at $T^* \in (0, E]$.

**Proof.** See the Appendix A.

For any given $A(\ge 0)$ and $T > 0$, calculate the first order partial derivative of $\Pi_1(A, p, T)$ with respect to $p$, we have:

$$\frac{\partial \Pi_1(A, p, T)}{\partial p} = \frac{1}{T}\left[\begin{array}{l} -b(A+1)^{\gamma} XT + (A+1)^{\gamma}(a-bp)T \\ +b(A+1)^{\gamma}\left(Y(1+E)\ln\left(\dfrac{1+E}{1+E-T}\right) + g\dfrac{T^2}{4} + h\left\{\dfrac{T^3}{9} - \dfrac{(1+E)T^2}{12}\right\}\right)\end{array}\right]. \tag{16}$$

The necessary condition to find the optimal selling price per unit $p^*$, by setting the first order partial derivative with respect to $p$ equals to zero, is:

$$(a-bp) - bX + b\left(Y\frac{(1+E)}{T}\ln\left(\frac{1+E}{1+E-T}\right) + g\frac{T}{4} + h\left\{\frac{T^2}{9} - \frac{(1+E)T}{12}\right\}\right) = 0. \tag{17}$$

This gives:

$$p^* = \frac{1}{2}\left[\frac{a}{b} - \left(\frac{(1+E)g}{2} + \frac{(1+E)^2 h}{6}\right) + Y\frac{(1+E)}{T}\ln\left(\frac{1+E}{1+E-T}\right) + g\frac{T}{4} + h\left\{\frac{T^2}{9} - \frac{(1+E)T}{12}\right\}\right]. \tag{18}$$

**Theorem 2.** For a given value of $A \geq 0$ and $T$>0; the total profit per unit time $\Pi_1(A, p, T)$ is a concave function in $p$ and hence there exists a unique $p^* \in \left(\frac{a}{2b}, \frac{a}{b}\right]$ from Eq. (18) such that $\Pi_1(A, p, T)$ is maximized; otherwise, the optimal selling price is $p^* = \frac{a}{b}$.

**Proof.** See the Appendix B.

Now the optimality of the frequency of advertisement $A$ is explored. Since $A$ is an integer number, the differential calculus is ineffectual to attain the optimal frequency of advertisement $A^*$. Additionally, the objective function $\Pi_1(A, p, T)$, for any given $p, T > 0$, is a concave function in $A$ over the interval $[0, \infty)$; otherwise, the objective function $\Pi_1(A, p, T)$ is maximized at $A^* = 0$ (see **Appendix C**). Hence, the optimal frequency of advertisement $A^*$, when $\Pi_1(A, p, T)$ is concave in $A$ over the interval $[0, \infty)$, can be found by comparing the values of $\Pi_1(\lfloor A^* \rfloor, p, T)$ and $\Pi_1(\lceil A^* \rceil, p, T)$ where $\lfloor u \rfloor = \max\{v : v \leq u$ and $v$ is integer$\}$ and $\lceil u \rceil = \min\{v$: $v \geq u$ and $v$ is integer$\}$. It is noteworthy point that, if $\Pi_1(\lfloor A^* \rfloor, p, T) = \Pi_1(\lceil A^* \rceil, p, T)$, there are two optimal solutions; otherwise, a unique solution exists (see, for instance, García-Laguna et al. (2010)).

Taking into account the concavity of the objective function $\Pi_1(A, p, T)$ in $A$ over the set of integer $N = \{0,1,2,3,...\}$, $A_l^* = \min\{A \in N : \Pi_1(A+1, p, T) \leq \Pi_1(A, p, T)\}$ provides the optimal solution if it is unique and the lower of them when two optimal solutions exist. Likewise, $A_u^* = \max\{A \in N : \Pi_1(A, p, T) \geq \Pi_1(A-1, p, T)\}$ provides the optimal solution if it is unique and the upper of them when two optimal solutions exist. The aforementioned conditions are equivalent to:

$$A_l^* = \min\left\{A \in N : (A+2)^\gamma - (A+1)^\gamma \leq \frac{G}{(a-bp)\psi_1}\right\}, \tag{19}$$

and $$A_u^* = \max\left\{A \in N : (A+1)^\gamma - A^\gamma \leq \frac{G}{(a-bp)\psi_1}\right\}, \tag{20}$$

respectively, where $\psi_1 = XT - \left(Y(1+E)\ln\left|\frac{1+E}{1+E-T}\right| + g\frac{T^2}{4} + h\left\{\frac{T^3}{9} - \frac{(1+E)T^2}{12}\right\}\right)$.

These two expressions are the necessary and sufficient conditions for the optimal frequency of advertisement when it is confined to be an integer. If the expression for $A_l^*$ is considered by taking into account the unique solution $U$ of the equation $(A+2)^{\gamma}-(A+1)^{\gamma}=\dfrac{G}{(a-bp)\psi_1}$, the optimal $A_l^*$ can be revealed as follows: $A_l^*=\lceil U\rceil$. Similarly, if the expression for $A_u^*$ is considered by taking into account the unique solution $V$ of the equation $(A+1)^{\gamma}-A^{\gamma}=\dfrac{G}{(a-bp)\psi_1}$, the optimal $A_u^*$ can be revealed as follows: $A_l^*=\lfloor V\rfloor$. It is noteworthy that the solutions $U$ and $V$ of the aforementioned equations are not possible to find out in the closed form. In order to find the values of $U$ and $V$, the equations are solved by using MATHEMATICA. Additionally, the solutions $U$ and $V$ must satisfy the equation $\lceil t\rceil=\lfloor t+1\rfloor$ which is true if and only if $t$ is not an integer. Consequently, a unique optimal $\left(A_l^*=A_u^*\right)$ frequency of advertisement is possible if and only if $U$ and $V$ are not an integer. Otherwise, two optimal solutions for $A$ are possible, namely, $A_l^*$ and $A_u^*=A_l^*+1$. Eventually, to find the optimal frequency of advertisement $A^*$ only one value of either $A_l^*$ from $(A+2)^{\gamma}-(A+1)^{\gamma}=\dfrac{G}{(a-bp)\psi_1}$ or $A_u^*$ from $(A+1)^{\gamma}-A^{\gamma}=\dfrac{G}{(a-bp)\psi_1}$ is necessary to be calculated.

Taking into account Theorem 1 and Theorem 2, the following algorithm is developed for finding the optimal solution of the inventory model without shortages.

**Algorithm for the optimal solution of the inventory model without shortage**

**Step 1.** Input the inventory parameters $C_0,G,C_p,a,b,g,h,E,\alpha,\gamma,L,n,I_c$ and initialize $T=\dfrac{E}{2}\in(0,E]$ and $p=\dfrac{0.8a}{b}\in\left(\dfrac{a}{2b},\dfrac{a}{b}\right)$.

**Step 2.** Set $i=1$ and $A^{(i)}=A=0$.

**Step 3.** Using the values of $T$ and $p$, calculate the solution $U$ of the equation $(A+2)^{\gamma}-(A+1)^{\gamma}=\dfrac{G}{(a-bp)\psi_1}$.

**Step 4.** If $U$ is not an integer, set $A^{(i+1)}=A_l^*=\lceil U\rceil$ then solve Eq. (14) and Eq. (17) for $T$ and $p$ with the value of $A=A^{(i+1)}$. Otherwise, go to Step 8.

**Step 5.** If $A^{(i+1)}\neq A^{(i)}$, set $i=i+1$ and go to **Step 3**. Otherwise, go to **Step 6**.

**Step 6.** Compute the optimal profit $\Pi_1^*(A, p, T)$ with the values of $p, T$ and $A = A^{(i+1)}$.

**Step 7.** Print the optimal solution: $A^*$, $p^*$, $T^*$ and $\Pi_1^*(A^*, p^*, T^*)$. Go to **Step 15.**

**Step 8.** Set $A_1^{(i+1)} = A_l^* = \lceil U \rceil$ and $A_2^{(i+1)} = A_l^* + 1$.

**Step 9.** If $A^{(i)} = A_1^{(i+1)} \neq A_2^{(i+1)}$ or $A^{(i)} = A_2^{(i+1)} \neq A_1^{(i+1)}$, there are two optimal solutions of $A$, say, $A_1^* = A_1^{(i+1)}$ and $A_2^* = A_2^{(i+1)}$. Go to **Step 14**.

**Step 10.** If $A^{(i)} \neq A_1^{(i+1)} \neq A_2^{(i+1)}$, solve Eq. (14) and Eq. (17) for $T_j$ and $p_j$ by dint of $A = A_j^{(i+1)}$ for $j = 1,2$. Set $i = i + 1$ and calculate the solutions $U_j$ of $(A+2)^\gamma - (A+1)^\gamma = \dfrac{G}{(a-bp)\psi_1}$ using $T_j$, $p_j$ for $j = 1,2$.

**Step 11.** If each $U_j$ is not an integer, set $A_j^{(i+1)} = \lceil U_j \rceil$ where $j = 1, 2$.

(a) If each $A_j^{(i+1)} = A_j^{(i)}$, there are two optimal solutions of $A$, say, $A_1^* = A_1^{(i+1)}$ and $A_2^* = A_2^{(i+1)}$. Go to **Step 14**.

(b) If $A_1^{(i+1)} \neq A_1^{(i)}$ or $A_2^{(i+1)} \neq A_2^{(i)}$ or both $A_1^{(i+1)} \neq A_1^{(i)}$ and $A_2^{(i+1)} \neq A_2^{(i)}$, solve Eq. (14) and Eq. (17) for $T_j$ and $p_j$ by dint of $A = A_j^{(i+1)}$ and then compute $\Pi_1^{(i)}\left(A_j^{(i+1)}, p_j, T_j\right)$ for $j = 1, 2$. Find the $A_j^{(i+1)}$ for which $\Pi_1^{(i)}\left(A_j^{(i+1)}, p_j, T_j\right)$ is larger between $\Pi_1^{(1)}\left(A_1^{(i+1)}, p_1, T_1\right)$ and $\Pi_1^{(2)}\left(A_2^{(i+1)}, p_2, T_2\right)$. Set $A^{(i+1)} = A_j^{(i+1)}$ and go to **Step 3**.

**Step 12.** If one or both of $U_j\,(j = 1,2)$ are not integers, set $A_j^{(i+1)} = \lceil U_j \rceil$ where $j = 1,2$. When $A_1^{(i+1)} = A_2^{(i+1)}$ or $A_1^{(i+1)} \sim A_2^{(i+1)} = 1$, solve Eq. (14) and Eq. (17) for $T$ and $p$ by dint of $A = A_1^{(i+1)}$. Set $A^{(i+1)} = A_1^{(i+1)}$ and go to **Step 3**. Otherwise, go to the next step.

**Step 13.** Solve Eq. (14) and Eq. (17) for $T_j$ and $p_j$ by dint of $A = A_j^{(i+1)}$ and then compute $\Pi_1^{(i)}\left(A_j^{(i+1)}, p_j, T_j\right)$ for $j = 1,2$. Find the $A_j^{(i+1)}$ for which $\Pi_1^{(i)}\left(A_j^{(i+1)}, p_j, T_j\right)$ is larger between $\Pi_1^{(1)}\left(A_1^{(i+1)}, p_1, T_1\right)$ and $\Pi_1^{(2)}\left(A_2^{(i+1)}, p_2, T_2\right)$. Set $A^{(i+1)} = A_j^{(i+1)}$ and go to **Step 3**.

**Step 14.** Solve Eq. (14) and Eq. (17) for $T_j$ and $p_j$ with the value of $A = A_j^*$ for $j = 1,2$. Print the optimal solutions: $\left(A_1^*, p_1^*, T_1^*\right)$ and $\left(A_2^*, p_2^*, T_2^*\right)$ with $\Pi_1^*\left(A_1^*, p_1^*, T_1^*\right) = \Pi_1^*\left(A_2^*, p_2^*, T_2^*\right)$.

**Step 15.** End.

*4.2. The inventory model with shortage*

Here, the concavity of the objective function $\Pi_2(A, p, t_1, t_2)$ in $p, t_1$ and $t_2$ for a fixed value of $A$ is examined in two folds. Firstly, the joint concavity of $\Pi_2(A, p, t_1, t_2)$ in $(t_1, t_2)$ is investigated for fixed values of $A$ and $p$. Secondly, the concavity of $\Pi_2(A, p, t_1, t_2)$ in $p$ for a fixed $A$ is explored at the optimal values $(t_1, t_2)$.

**Theorem 3.** For any given values of $A \geq 0$ and $p > 0$; $\Pi_2(A, p, t_1, t_2)$ is a pseudo-concave function of $t_1$ and $t_2$, and therefore there exists a unique pair of values $(t_1, t_2)$ such that $\Pi_2(A, p, t_1, t_2)$ achieves its global maximum value.

**Proof.** See the Appendix D.

To obtain the optimal stock-in period $t_1^*$, optimal stock-out period $t_2^*$, for any fixed $A$ and $p$, calculate the first order partial derivatives of $\Pi_2(A, p, t_1, t_2)$ with respect to $t_1$ and $t_2$.

$$\frac{\partial \Pi_2(\cdot)}{\partial t_1} = -\frac{\Pi_2(\cdot)}{t_1 + t_2} + \frac{(A+1)^\gamma (a - bp)}{t_1 + t_2}\left[X - Y\frac{1+E}{1+E-t_1} - g\frac{t_1}{2} - h\left\{\frac{t_1^{\,2}}{3} - \frac{(1+E)t_1}{6}\right\}\right], \tag{21}$$

and

$$\frac{\partial \Pi_2(\cdot)}{\partial t_2} = -\frac{\Pi_2(\cdot)}{t_1 + t_2} + \frac{(A+1)^\gamma (a - bp)}{t_1 + t_2}\left[X_1\frac{1}{1+\delta t_2} - \left(C_l + \frac{C_s}{\delta}\right)\right]. \tag{22}$$

Therefore, the necessary conditions for the optimal values of $t_1$ and $t_2$ are:

$$\frac{\partial \Pi_2(\cdot)}{\partial t_1} = 0, \tag{23}$$

and

$$\frac{\partial \Pi_2(\cdot)}{\partial t_2} = 0. \tag{24}$$

By dint of Eq. (23) and Eq. (24), one has

$$X - Y\frac{1+E}{1+E-t_1} - g\frac{t_1}{2} - h\left[\frac{t_1^{\,2}}{3} - \frac{(1+E)t_1}{6}\right] = X_1\frac{1}{1+\delta t_2} - \left(C_l + \frac{C_s}{\delta}\right). \tag{25}$$

This gives:

$$t_2^{\ *}=\frac{1}{\delta}\left[\frac{X_1}{X-Y\dfrac{1+E}{1+E-t_1}-g\dfrac{t_1}{2}-h\left[\dfrac{t_1^{\ 2}}{3}-\dfrac{(1+E)t_1}{6}\right]+\left(C_l+\dfrac{C_s}{\delta}\right)}-1\right]. \tag{26}$$

Eq. (26) reveals that the optimal shortages period can be found with the help of the values of $A,\ t_1$ and $p$. On simplification, Eq. (23) can be written as:

$$\begin{aligned}&-\frac{C_0+GA}{(A+1)^{\gamma}(a-bp)}+Y(1+E)\left\{\frac{t_1}{1+E-t_1}-\ln\left|\frac{1+E}{1+E-t_1}\right|\right\}+g\frac{t_1^{\ 2}}{4}+h\left[\frac{2t_1^{\ 3}}{9}-\frac{(1+E)t_1^{\ 2}}{12}\right]\\&+\frac{1}{\delta}\left[X_1\ln\left|F(t_1)\right|-Y\frac{t_1}{1+E-t_1}-g\frac{t_1}{2}-h\left\{\frac{t_1^{\ 2}}{3}-\frac{(1+E)t_1}{6}\right\}\right]=0,\end{aligned} \tag{27}$$

where $$F(t_1)=\frac{X_1}{X-Y\dfrac{1+E}{1+E-t_1}-g\dfrac{t_1}{2}-h\left[\dfrac{t_1^{\ 2}}{3}-\dfrac{(1+E)t_1}{6}\right]+\left(C_l+\dfrac{C_s}{\delta}\right)}. \tag{28}$$

For any fixed value of $A$ and $p$, Eq. (27) consists of only one decision variable i.e., $t_1$. Hence, the optimal period of positive on-hand inventory level ($t_1^{\ *}$) can be easily obtained by solving Eq. (27) for $t_1$.

To investigate the existence of a unique value of $t_1\in(0,E]$ at which $\Pi_2(A,p,t_1,t_2)$ is maximized, for convenience, define:

$$\begin{aligned}\Delta_2=&\frac{C_0+GA}{(A+1)^{\gamma}(a-bp)}+Y(1+E)\ln\left|1+E\right|+g\frac{E^2}{4}+h\left[\frac{E^3}{9}-\frac{(1+E)E^2}{12}\right]\\&-X_1\frac{\ln\left|F(E)\right|}{\delta}+X_1\frac{\delta E-1+F(E)}{\delta F(E)}-\left(X+C_l+\frac{C_s}{\delta}\right)E.\end{aligned} \tag{29}$$

So, the following theorem is proposed.

**Theorem 4.** For any given values of $p>0$ and $A\geq 0$;

(a) If $\Delta_2>0$, then the total profit per unit time $\Pi_2(A,p,t_1,t_2)$ attains its global maximum value at $\left(t_1^*,t_2^*\right)=\left(E,\dfrac{1}{\delta}\{F(E)-1\}\right)$.

(b) If $\Delta_2 \le 0$, then total profit per unit time $\Pi_2(A, p, t_1, t_2)$ achieves its global maximum value at $(t_1^*, t_2^*)$ where $t_1^* \in (0, E)$ and $t_2^*$ satisfy Eq. (27) and Eq. (26) respectively.

**Proof.** See the Appendix E.

For a given value of $A$, $t_1$ and $t_2$, the first order partial derivatives of $\Pi_2(A, p, t_1, t_2)$ with respect to $p$ is expressed below as:

$$\frac{\partial \Pi_2(\cdot)}{\partial p} = \frac{(A+1)^{\gamma}}{t_1 + t_2}\left[\begin{array}{l} (a-bp)t_1 - bXt_1 + bY(1+E)\ln\left|\dfrac{1+E}{1+E-t_1}\right| + gb\dfrac{t_1^{\,2}}{4} + hb\left[\dfrac{t_1^{\,3}}{9} - \dfrac{(1+E)t_1^{\,2}}{12}\right] \\ +(a-bp)\dfrac{\ln|1+\delta t_2|}{\delta} - bX_1\dfrac{\ln|1+\delta t_2|}{\delta} + \left(C_l + \dfrac{C_s}{\delta}\right)bt_2 \end{array}\right]. \qquad (30)$$

The necessary condition for the optimal selling price $p^*$ is when the first order partial derivative with respect to $p$ is equal to zero, thus:

$$\begin{array}{l} (a-bp)\left[t_1 + \dfrac{\ln|1+\delta t_2|}{\delta}\right] - bXt_1 + bY(1+E)\ln\left|\dfrac{1+E}{1+E-t_1}\right| + bg\dfrac{t_1^{\,2}}{4} + bh\left[\dfrac{t_1^{\,3}}{9} - \dfrac{(1+E)t_1^{\,2}}{12}\right] \\ -bX_1\dfrac{\ln|1+\delta t_2|}{\delta} + \left(C_l + \dfrac{C_s}{\delta}\right)bt_2 = 0. \end{array} \qquad (31)$$

The optimal selling price per unit $(p^*)$ from Eq. (31) is:

$$p^* = \frac{\delta}{2(\delta t_1 + \ln|1+\delta t_2|)}\left[\begin{array}{l} \dfrac{a}{b}\left[t_1 + \dfrac{\ln|1+\delta t_2|}{\delta}\right] - X_2 t_1 + Y(1+E)\ln\left|\dfrac{1+E}{1+E-t_1}\right| + g\dfrac{t_1^{\,2}}{4} \\ +h\left[\dfrac{t_1^{\,3}}{9} - \dfrac{(1+E)t_1^{\,2}}{12}\right] - X_3\dfrac{\ln|1+\delta t_2|}{\delta} + \left(C_l + \dfrac{C_s}{\delta}\right)t_2 \end{array}\right], \qquad (32)$$

where $X_2 = \frac{g}{2}(1+E) + \frac{h}{6}(1+E)^2$ and $X_3 = C_l + \frac{C_s}{\delta} - \left(1 + \frac{n+1}{2n} I_c \alpha L\right)C_p$.

**Theorem 5.** For the given value of $A, t_1$ and $t_2$; the $\Pi_2(A, p, t_1, t_2)$ is a concave function in $p$ and therefore there exists a unique $p^* \in \left(\frac{a}{2b}, \frac{a}{b}\right]$ from Eq. (32) such that $\Pi_2(A, p, t_1, t_2)$ is maximized; otherwise, the optimal selling price is $p^* = \frac{a}{b}$.

**Proof.** See the Appendix F.

Now the optimality of the frequency of advertisement $A$ is examined by analysis similar to the previous section.

Taking into account the concavity of the objective function $\Pi_2(A, p, t_1, t_2)$ in $A$ over the set of the integers $N = \{0,1,2,3,...\}$ , $A_l^* = \min\left\{A \in N : (A+2)^\gamma - (A+1)^\gamma \le \frac{G}{(a-bp)\psi_2}\right\}$ delivers the optimal solution if it is unique and the lower of them when two optimal solutions exist. Likewise, $A_u^* = \max\left\{A \in N : (A+1)^\gamma - A^\gamma \le \frac{G}{(a-bp)\psi_2}\right\}$ supplies the optimal solution if it is unique and the upper of them when two optimal solutions exist where $\psi_2 = Xt_1 - Y(1+E)\ln\left|\frac{1+E}{1+E-t_1}\right| - g\frac{t_1^2}{4} - h\left[\frac{t_1^3}{9} - \frac{(1+E)t_1^2}{12}\right] + X_1\frac{\ln|1+\delta t_2|}{\delta} - \left(C_l + \frac{C_s}{\delta}\right)t_2$.

The two expressions before mentioned are the necessary and sufficient conditions for the optimal frequency of advertisement when it is restricted to be an integer. If the expression for $A_l^*$ is considered by taking into account the unique solution $U$ of the equation $(A+2)^\gamma - (A+1)^\gamma = \frac{G}{(a-bp)\psi_2}$, the optimal $A_l^*$ can be revealed as: $A_l^* = \lceil U \rceil$. Similarly, if the expression for $A_u^*$ is considered by taking into account the unique solution $V$ of the equation $(A+1)^\gamma - A^\gamma = \frac{G}{(a-bp)\psi_2}$, the optimal $A_u^*$ can be revealed as: $A_l^* = \lfloor V \rfloor$. It is remarkable that the solutions $U$ and $V$ of the above-mentioned equations are not possible to find out in closed form. In order to find the values of $U$ and $V$, the equations are solved by using MATHEMATICA.

The following algorithm is developed using Theorem 3, Theorem 4and Theorem 5. This algorithm determines the optimal solution for the inventory model with partial backlogging.

**Algorithm for determining the optimal solution of the inventory model with shortage**

**Step 1.** Input the inventory parameters $C_0, G, C_p, a, b, g, h, E, \alpha, \gamma, L, n, I_c, \delta, C_l, C_s$ and initialize $t_1 = \frac{E}{2} \in (0, E]$, $t_2 = 0$ and $p = \frac{0.8a}{b} \in \left(\frac{a}{2b}, \frac{a}{b}\right)$.

**Step 2.** Set $i = 1$ and $A^{(i)} = A = 0$.

**Step 3.** Using the values of $t_1, t_2$ and $p$ , calculate the solution $U$ of the equation

$$(A+2)^{\gamma}-(A+1)^{\gamma}=\frac{G}{(a-bp)\psi_2}.$$

**Step 4.** If $U$ is not an integer, set $A^{(i+1)}=A_l^*=\lceil U\rceil$ and solve Eq. (27) and Eq. (31) by putting $A=A^{(i+1)}$ for $t_1$ and $p$ with the help of Eq. (26). Calculate the corresponding $t_2$ from Eq. (26); go to the next step. Otherwise, go to Step 8.

**Step 5.** If $A^{(i+1)}\neq A^{(i)}$, set $i=i+1$ and go to **Step 3**. Otherwise, go to **Step 6**.

**Step 6.** Compute the optimal profit $\Pi_2^*(A,p,t_1,t_2)$ with the values of $p,t_1,t_2$ and $A=A^{(i+1)}$.

**Step 7.** Print the optimal solution: $A^*$, $p^*$, $t_1^*$, $t_2^*$ and $\Pi_2^*(A^*,p^*,t_1^*,t_2^*)$. Go to **Step 15.**

**Step 8.** Set $A_1^{(i+1)}=A_l^*=\lceil U\rceil$ and $A_2^{(i+1)}=A_l^*+1$.

**Step 9.** If $A^{(i)}=A_1^{(i+1)}\neq A_2^{(i+1)}$ or $A^{(i)}=A_2^{(i+1)}\neq A_1^{(i+1)}$, there are two optimal solutions of $A$, say, $A_1^*=A_1^{(i+1)}$ and $A_2^*=A_2^{(i+1)}$. Go to **Step 14**.

**Step 10.** If $A^{(i)}\neq A_1^{(i+1)}\neq A_2^{(i+1)}$, solve Eq. (27) and Eq. (31) by putting $A=A^{(i+1)}$ for $t_1^{(j)}$ and $p^{(j)}$ with the help of Eq. (26). Calculate the corresponding $t_2^{(j)}$ from Eq. (26). Set $i=i+1$ and calculate the solutions $U_j$ of $(A+2)^{\gamma}-(A+1)^{\gamma}=\dfrac{G}{(a-bp)\psi_1}$ using $t_1^{(j)},t_2^{(j)}$ and $p^{(j)}$ for $j=1,2$.

**Step 11.** If each $U_j$ is not an integer, set $A_j^{(i+1)}=\lceil U_j\rceil$ where $j=1,2$.

(a) If each $A_j^{(i+1)}=A_j^{(i)}$, there are two optimal solutions of $A$, say, $A_1^*=A_1^{(i+1)}$ and $A_2^*=A_2^{(i+1)}$. Go to **Step 14**.

(b) If $A_1^{(i+1)}\neq A_1^{(i)}$ or $A_2^{(i+1)}\neq A_2^{(i)}$ or both $A_1^{(i+1)}\neq A_1^{(i)}$ and $A_2^{(i+1)}\neq A_2^{(i)}$, solve Eq. (27) and Eq. (31) by putting $A=A_j^{(i+1)}$ for $t_1^{(j)}$ and $p^{(j)}$ with the help of Eq. (26). Calculate the corresponding $t_2^{(j)}$ from Eq. (26) and then compute $\Pi_2^{(i)}\left(A_j^{(i+1)},p^{(j)},t_1^{(j)},t_2^{(j)}\right)$ for $j=1,2$. Find the $A_j^{(i+1)}$ for which $\Pi_2^{(i)}\left(A_j^{(i+1)},p^{(j)},t_1^{(j)},t_2^{(j)}\right)$ is larger between $\Pi_2^{(1)}\left(A_1^{(i+1)},p^{(1)},t_1^{(1)},t_2^{(1)}\right)$ and $\Pi_2^{(2)}\left(A_2^{(i+1)},p^{(2)},t_1^{(2)},t_2^{(2)}\right)$. Set $A^{(i+1)}=A_j^{(i+1)}$ and go to **Step 3**.

**Step 12.** If one or both of $U_j\,(j=1,2)$ are not integers, set $A_j^{(i+1)}=\lceil U_j\rceil$ where $j=1,2$ .When

$A_1^{(i+1)} = A_2^{(i+1)}$ or $A_1^{(i+1)} \sim A_2^{(i+1)} = 1$, solve Eq. (27) and Eq. (31) by putting $A = A_1^{(i+1)}$ for $t_1$ and $p$ with the help of Eq. (26). Calculate the corresponding $t_2$ from Eq. (26). Set $A^{(i+1)} = A_1^{(i+1)}$ and go to **Step 3**. Otherwise, go to the next step.

**Step 13.** Solve Eq. (27) and Eq. (31) by putting $A = A_j^{(i+1)}$ for $t_1^{(j)}$ and $p^{(j)}$ with the help of Eq. (26). Calculate the corresponding $t_2^{(j)}$ from Eq. (26) and then compute $\Pi_2^{(i)}\left(A_j^{(i+1)}, p^{(j)}, t_1^{(j)}, t_2^{(j)}\right)$ for $j = 1,2$ . Find the $A_j^{(i+1)}$ for which $\Pi_2^{(i)}\left(A_j^{(i+1)}, p^{(j)}, t_1^{(j)}, t_2^{(j)}\right)$ is larger between $\Pi_2^{(1)}\left(A_1^{(i+1)}, p^{(1)}, t_1^{(1)}, t_2^{(1)}\right)$ and $\Pi_2^{(2)}\left(A_2^{(i+1)}, p^{(2)}, t_1^{(2)}, t_2^{(2)}\right)$. Set $A^{(i+1)} = A_j^{(i+1)}$ and go to **Step 3**.

**Step 14.** Solve Eq. (27) and Eq. (31) by putting $A = A_j^{*}$ for $t_1^{(j)}$ and $p^{(j)}$ with the help of Eq. (26) where $j = 1,2$ . Calculate the corresponding $t_2^{(j)}$ from Eq. (26). Print the optimal solutions: $\left(A_1^{*}, p^{(1)*}, t_1^{(1)*}, t_2^{(1)*}\right)$ and $\left(A_2^{*}, p^{(2)*}, t_1^{(2)*}, t_2^{(2)*}\right)$ with $\Pi_2^{*}\left(A_1^{*}, p^{(1)*}, t_1^{(1)*}, t_2^{(1)*}\right) = \Pi_2^{*}\left(A_2^{*}, p^{(2)*}, t_1^{(2)*}, t_2^{(2)*}\right)$ .

**Step 15.** End.

## 5. Numerical illustration

This section solves three numerical examples. The first two examples illustrate the inventory model without shortages and the second one exemplifies the inventory model with shortages.

*Example 1. The inventory model without shortages*

Let $C_0 = \$520$ /order, $a = 100$ , $b = 1.5$ , $C_p = \$5$ /unit, $g = \$1$ /unit/week, $h = \$0.25$ /unit/(week)$^2$,$E$=4weeks, $L$=5weeks, $n = 3$, $I_c = 0.05$/week, $\alpha$=0.4, $\gamma = 0.1$, $G = \$50$/advertisement. The optimal solution can be obtained by dint of the algorithm of the inventory model without shortages in section 4.1 as follows:

Step 1: Initialize $T = \dfrac{E}{2}$ and $p = \dfrac{0.8a}{b}$.

Iteration 1:

Step 2: Set $i = 1$ and $A^{(i)} = A = 0$.

Step 3: The solution $U$ of the equation $(A+2)^{\gamma} - (A+1)^{\gamma} = \dfrac{G}{(a-bp)\psi_1}$ is 2.6838.

Step4: Since $U$ is not an integer, set $A^{(i+1)} = A_l^* = \lceil 2.6838 \rceil = 3$. Solving Eq. (14) and Eq. (17) for $T$ and $p$ with the value of $A = 3$ one can obtain $T$=2.396928 and $p$=37.85930.

Step5: Since $A^{(i+1)} \neq A^{(i)}$, set $i = i + 1$ and go to **Step 3**.

Iteration 2:

Step 3: The solution $U$ of the equation $(A+2)^{\gamma} - (A+1)^{\gamma} = \dfrac{G}{(a-bp)\psi_1}$ is 5.78836.

Step 4: Since $U$ is not an integer, set $A^{(i+1)} = A_l^* = \lceil 5.78836 \rceil = 6$ .Solving Eq. (14) and Eq. (17) for $T$ and $p$ with the value of $A$=6 one can obtain $T$=2.515376 and $p$=38.00229.

Step 5: Since $A^{(i+1)} \neq A^{(i)}$, set $i = i + 1$ and go to **Step 3**.

Iteration 3:

Step 3: The solution $U$ of the equation $(A+2)^{\gamma} - (A+1)^{\gamma} = \dfrac{G}{(a-bp)\psi_1}$ is 6.10415.

Step 4: Since $U$ is not an integer, set $A^{(i+1)} = A_l^* = \lceil 6.10415 \rceil = 7$ .Solving Eq. (14) and Eq. (17) for $T$ and $p$ with the value of $A$=7 one can obtain $T$=2.552968 and $p$=38.04934.

Step 5: Since $A^{(i+1)} \neq A^{(i)}$, set $i = i + 1$ and go to **Step 3**.

Iteration 4:

Step 3: The solution $U$ of the equation $(A+2)^{\gamma} - (A+1)^{\gamma} = \dfrac{G}{(a-bp)\psi_1}$ is 6.20208.

Step 4: Since $U$ is not an integer, set $A^{(i+1)} = A_l^* = \lceil 6.20208 \rceil = 7$ .Solving Eq. (14) and Eq. (17) for $T$ and $p$ with the value of $A$=7 one can obtain $T$=2.552968 and $p$=38.10817.

Step 5: As $A^{(i+1)} = A^{(i)} = 7$, go to **Step 6**.

Step 6: The optimal total profit per unit time using $A$=7, $p$=38.04934 and $T$=2.552968 is $\Pi_1^*(A, p, T) = 1171.591$.

Step 7: The optimal solution: $A^* = 7$, $p^* = 38.049$, $T^* = 2.553$ and $\Pi_1^*(A^*, p^*, T^*) = 1171.591$.

Therefore, the optimal solution is given by $A^* = 7$ , $p^* = \$38.049$ , $T^* = 2.553$ weeks, $Q^* = 188.816$ units and the total profit per unit time is $\Pi_1^{(\max)}(A, p, T) = \$1171.591$/week. The concavity of the profit function for Example 1 is observed in the Figure3for a fixed value of the frequency of advertisement and also the location of the optimal solution can be identified by the point with magenta color.

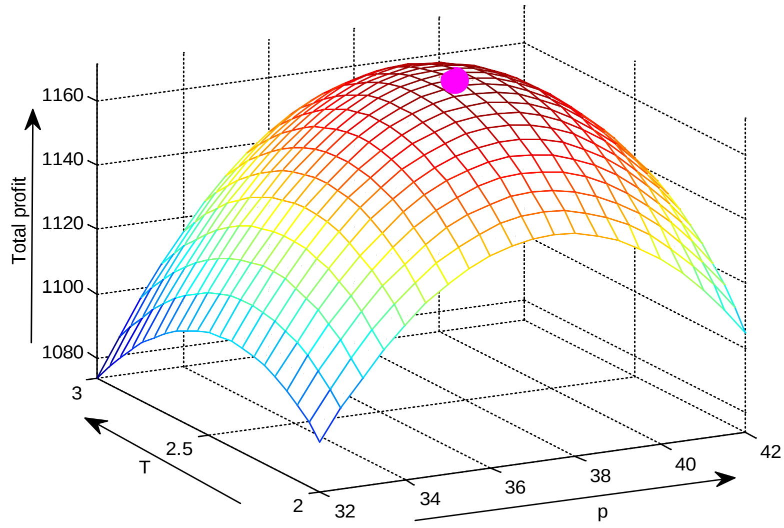


**Figure3.** The concavity of the profit function $\Pi_1(A, p, T)$ for $A = 7$.

*Example 2. The inventory model without shortage*

Let $C_0 = \$520$ /order, $a = 100$ , $b = 2.5$ , $C_p = \$15$ /unit, $g = \$2$ /unit/week, $h = \$0.25$ /unit/(week)$^2$, *E*=4weeks, *L*=5weeks, $n = 3$, $I_c = 0.05$ /week, *α*=0.4, $\gamma = 0.1$, $G = \$50$/advertisement. The algorithm of the inventory model without shortages in section 4.1 provides the optimal solution of the problem as follows:

Step 1: Initialize $T = \dfrac{E}{2}$ and $p = \dfrac{0.8a}{b}$.

Iteration 1:

Step 2: Set $i = 1$ and $A^{(i)} = A = 0$.

Step 3: The solution $U$ of the equation $(A+2)^\gamma - (A+1)^\gamma = \dfrac{G}{(a-bp)\psi_1}$ is -0.695453.

Step 4: Since $U$ is not an integer, set $A^{(i+1)} = A_l^* = \lceil -0.695453 \rceil = 0$. Solving Eq. (14) and Eq. (17) for $T$ and $p$ with the value of *A*=0 one can obtain *T*=2.554998 and *p*=32.14957.

Step 5: As $A^{(i+1)} = A^{(i)} = 0$, go to **Step 6**.

Step 6: The optimal profit per unit time using $A=0$, $p=32.14957$ and $T=2.254998$ is $\Pi_1^*(A,p,T)=$ -98.43092.

Step 7: The optimal solution: $A^*=0$, $p^*=\$32.14957$, $T^*=2.254998$ and $\Pi_1^*(A^*,p^*,T^*)=$ -87.64017.

Thus, the optimal solution is given by $A^*=0$, $p^*=\$32.15$, $T^*=2.255$ weeks, $Q^*=63.068$ units and the total profit is $\Pi_1^{(\max)}(A,p,T)=$ -\$87.6402 / week. The inventory systems are not always profitable and the Example 2 reveals that the inventory system is not profitable in this case. Moreover, the optimal frequency of advertisement is zero which justifies the results found in Appendix C.

*Example 3.The inventory model with shortages*

For this example, it is considered the same data of the Example 1along with $\delta$=0.4, $C_s$=\$3/unit and $C_l$=\$6/unit. The optimal solution using the algorithm of the inventory model without shortages in section 4.2 can be computed as follows:

Step 1: Initialize $t_1=\frac{E}{2}$, $t_2=0$ and $p=\frac{0.8a}{b}$.

Iteration 1:

Step 2: Set $i=1$ and $A^{(i)}=A=0$.

Step 3: The solution $U$ of the equation $(A+2)^{\gamma}-(A+1)^{\gamma}=\frac{G}{(a-bp)\psi_2}$ is 2.6838.

Step 4: Since $U$ is not an integer, set $A^{(i+1)}=A_l^*=\lceil 2.6838 \rceil=3$. Solving Eq. (27) and Eq. (31) by putting $A$=3 for $t_1$ and $p$ with the help of Eq. (26) one can obtain $t_1$=2.251032, $p$=37.51684. The corresponding $t_2$, from Eq. (26), is 0.5588405.

Step 5: Since $A^{(i+1)}\neq A^{(i)}$, set $i=i+1$ and go to **Step 3**.

Iteration 2:

Step 3: The solution $U$ of the equation $(A+2)^{\gamma}-(A+1)^{\gamma}=\frac{G}{(a-bp)\psi_2}$ is 7.23158.

Step 4: Since $U$ is not an integer, set $A^{(i+1)}=A_l^*=\lceil 7.23158 \rceil=8$ .Solving Eq. (27) and Eq. (31) by putting $A$=8 for $t_1$ and $p$ with the help of Eq. (26) one can obtain $t_1$=2.429471, $p$=37.69551. The corresponding $t_2$, from Eq. (26), is 0.6608389.

Step 5: Since $A^{(i+1)}\neq A^{(i)}$, set $i=i+1$ and go to **Step 3**.

Iteration 3:

Step 3: The solution $U$ of the equation $(A+2)^{\gamma}-(A+1)^{\gamma}=\dfrac{G}{(a-bp)\psi_2}$ is 8.02072.

Step 4: Since $U$ is not an integer, set $A^{(i+1)}=A_l^*=\lceil 8.02072 \rceil = 9$. Solving Eq. (27) and Eq. (31) by putting $A$=9 for $t_1$ and $p$ with the help of Eq. (26) one can obtain $t_1$=2.461948, $p$=37.72961. The corresponding $t_2$, from Eq. (26), is 0.6815652.

Step 5: Since $A^{(i+1)} \neq A^{(i)}$, set $i=i+1$ and go to **Step 3**.

Iteration 4:

Step 3: The solution $U$ of the equation $(A+2)^{\gamma}-(A+1)^{\gamma}=\dfrac{G}{(a-bp)\psi_2}$ is 8.16652.

Step 4: Since $U$ is not an integer, set $A^{(i+1)}=A_l^*=\lceil 8.16652 \rceil = 9$. Solving Eq. (27) and Eq. (31) by putting $A$=9 for $t_1$ and $p$ with the help of Eq. (26) one can obtain $t_1$=2.461948, $p$=37.72961. The corresponding $t_2$, from Eq. (26), is 0.6815652.

Step 5: As $A^{(i+1)}=A^{(i)}=9$, go to **Step 6**.

Step 6: The optimal total profit per unit time using $A$=9, $t_1$=2.461948, $t_2$=0.6815652 and $p$=37.72961 is $\Pi_2^*(A,p,t_1,t_2)=1233.009$.

Step 7: The optimal solution: $A^*=9$, $t_1^*=2.461948$, $t_2^*=0.6815652$, $p^*=37.72961$ and $\Pi_2^*(A^*,p^*,t_1^*,t_2^*)=1233.009$.

Hence, the optimal solution is as follows: $A^*=9$, $p^*=\$37.730$, $t_1^*=2.462$ weeks, $t_2^*=0.682$ week, $T^*=3.144$ weeks, $S^*=185.256$ units, $R^*=32.935$ units, $Q^*=218.190$ units and $\Pi_2^{(\max)}(A,p,t_1,t_2)=\$1233.009$. The concavity of the total profit function is observed from the Figure4, for a fixed value of the frequency of advertisement and also the point with magenta color indicates the location of the optimal solution.

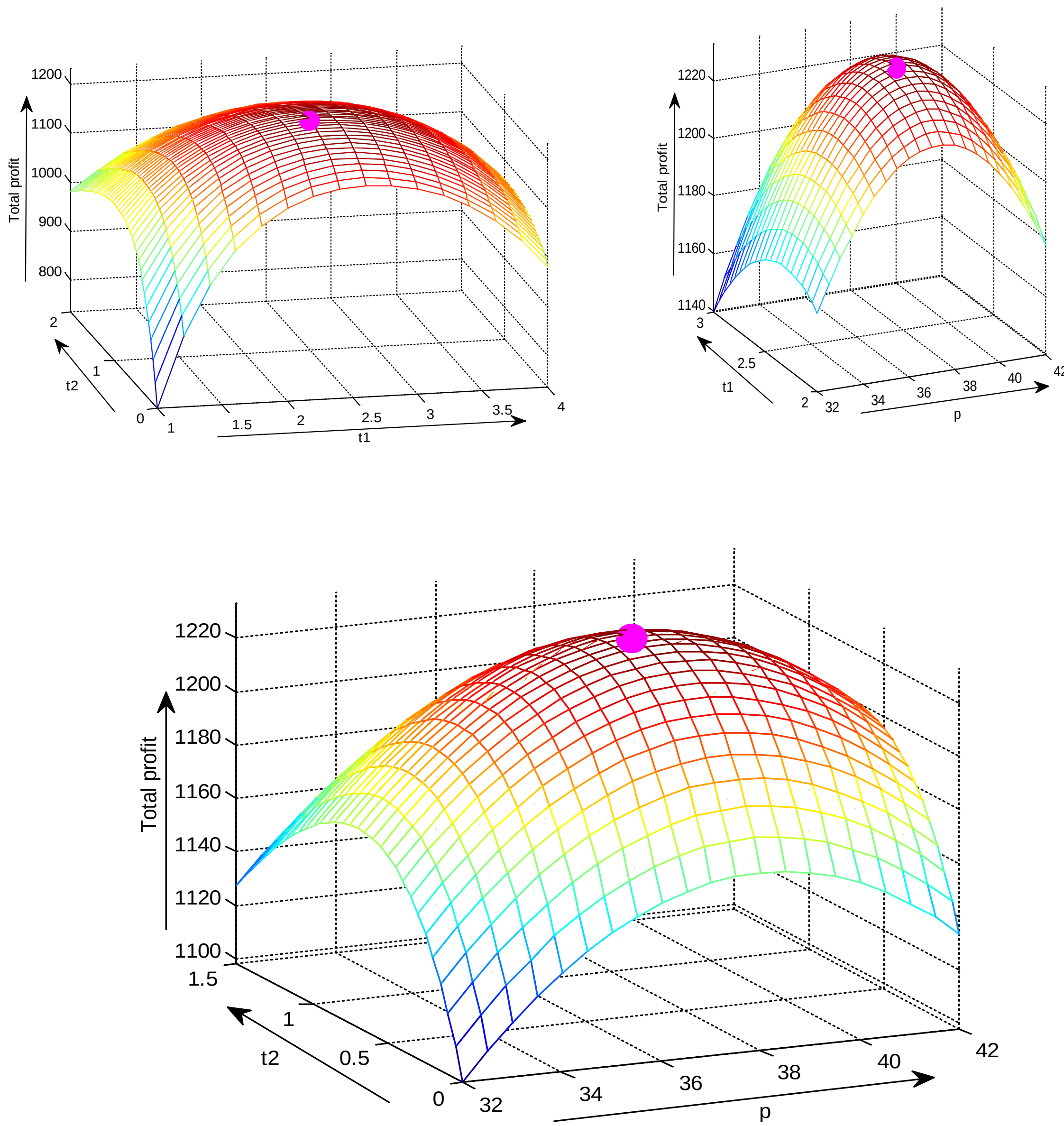


**Figure4.** The concavity of the profit functions $\Pi_2(A, t_1, p, t_2)$ for $A = 9$.

## 6. Sensitivity analysis

A sensitivity analysis is carried out in order to investigate the impact of changes of input parameters on optimal solution of $p, t_1, t_2, S, R$ and the total profit per unit time for the Example 3.The results are shown in Table 2.

**Table 2.** Sensitivity analysis of input parameters on the optimal solution of Example 3.

| Parameter | Original | % of | $A^*$ | %of change in |
|---|---|---|---|---|

| | value | change in the original values | | $p^*$ | $t_1^*$ | $t_2^*$ | $S^*$ | $R^*$ | Total profit |
|---|---|---|---|---|---|---|---|---|---|
| $C_0$ | 520 | -20% | 8 | -0.34 | -5.03 | -11.04 | -7.59 | -10.48 | 2.78 |
| | | -10% | 8 | -0.21 | -3.12 | -7.03 | -5.14 | -7.02 | 1.37 |
| | | 10% | 9 | 0.11 | 1.63 | 3.91 | 2.2 | 3.31 | -1.33 |
| | | 20% | 9 | 0.22 | 3.19 | 7.82 | 4.32 | 6.58 | -2.63 |
| $a$ | 100 | -20% | 5 | -17.53 | 1.48 | 27.17 | -25.49 | -9.92 | -49.34 |
| | | -10% | 7 | -8.78 | 0.62 | 11.83 | -12.77 | -4.57 | -26.68 |
| | | 10% | 11 | 8.8 | -0.48 | -9.47 | 12.84 | 3.96 | 30.79 |
| | | 20% | 14 | 17.69 | 0.22 | -15.18 | 28.51 | 10.54 | 65.76 |
| $b$ | 1.5 | -20% | 12 | 22.35 | 3.73 | -11.58 | 11.22 | -5.5 | 41.55 |
| | | -10% | 10 | 9.9 | 1.27 | -6.6 | 4.27 | -3.66 | 18.25 |
| | | 10% | 7 | -8.21 | -2.65 | 2.5 | -7.07 | -1.31 | -14.62 |
| | | 20% | 6 | -14.98 | -4.01 | 7.32 | -11.33 | 0.01 | -26.57 |
| $C_p$ | 5 | -20% | 9 | -1.63 | 3.5 | -6.7 | 7.32 | -4 | 6.13 |
| | | -10% | 9 | -0.81 | 1.69 | -3.36 | 3.52 | -1.97 | 3.03 |
| | | 10% | 8 | 0.72 | -2.9 | 0.23 | -5.98 | -1.77 | -2.95 |
| | | 20% | 8 | 1.52 | -4.38 | 3.53 | -8.96 | 0.02 | -5.83 |
| $g$ | 1 | -20% | 9 | -0.17 | 2.61 | -3.22 | 4.01 | -2.66 | 1.11 |
| | | -10% | 9 | -0.08 | 1.29 | -1.60 | 1.96 | -1.31 | 0.55 |
| | | 10% | 8 | -0.01 | -2.57 | -1.51 | -4.62 | -2.37 | -0.53 |
| | | 20% | 8 | 0.06 | -3.78 | -0.01 | -6.38 | -1.14 | -1.04 |
| $h$ | 0.25 | -20% | 9 | -0.01 | 0.83 | -0.62 | 1.2 | -0.54 | 0.21 |
| | | -10% | 9 | -0.01 | 0.41 | -0.31 | 0.59 | -0.27 | 0.11 |
| | | 10% | 9 | 0.005 | -0.4 | 0.31 | -0.58 | 0.27 | -0.11 |
| | | 20% | 9 | 0.01 | -0.8 | 0.61 | -1.15 | 0.53 | -0.21 |
| $C_s$ | 3 | -20% | 9 | -0.07 | -0.24 | 3.81 | -0.25 | 3.46 | 0.17 |
| | | -10% | 9 | -0.03 | -0.12 | 1.87 | -0.12 | 1.70 | 0.08 |
| | | 10% | 9 | 0.03 | 0.11 | -1.80 | 0.12 | -1.64 | -0.08 |
| | | 20% | 9 | 0.06 | 0.22 | -3.54 | 0.24 | -3.24 | -0.16 |
| $C_l$ | 6 | -20% | 9 | -0.05 | -0.19 | 3.03 | -0.2 | 2.75 | 0.14 |
| | | -10% | 9 | -0.03 | -0.09 | 1.49 | -0.1 | 1.36 | 0.07 |
| | | 10% | 9 | 0.03 | 0.09 | -1.45 | 0.1 | -1.32 | -0.07 |
| | | 20% | 9 | 0.05 | 0.18 | -2.85 | 0.19 | -2.61 | -0.13 |
| $\delta$ | 0.4 | -20% | 9 | -0.11 | -1.04 | 16.05 | -1.33 | 17.15 | 0.81 |
| | | -10% | 9 | -0.05 | -0.48 | 7.41 | -0.62 | 7.89 | 0.38 |
| | | 10% | 9 | 0.05 | 0.42 | -6.44 | 0.54 | -6.80 | -0.33 |
| | | 20% | 9 | 0.09 | 0.78 | -12.09 | 1.00 | -12.73 | -0.62 |
| $E$ | 4 | -20% | 8 | -0.02 | -10.52 | 3.91 | -8.85 | 2.4 | -2.36 |
| | | -10% | 8 | -0.06 | -5.71 | 0.15 | -5.62 | -0.84 | -1.1 |
| | | 10% | 9 | -0.03 | 4.1 | -2.86 | 2.57 | -2.51 | 0.98 |
| | | 20% | 9 | -0.06 | 7.86 | -5.33 | 4.82 | -4.69 | 1.85 |
| $n$ | 3 | -20% | 9 | 0.25 | -0.51 | 1.05 | -1.05 | 0.6 | -0.94 |
| | | -10% | 9 | 0.06 | -0.13 | 0.26 | -0.26 | 0.15 | -0.23 |
| | | 10% | 9 | -0.03 | 0.06 | -0.13 | 0.13 | -0.08 | 0.12 |
| | | 20% | 9 | -0.05 | 0.1 | -0.21 | 0.21 | -0.12 | 0.19 |
| $L$ | 5 | -20% | 9 | -0.1 | 0.21 | -0.42 | 0.43 | -0.24 | 0.38 |
| | | -10% | 9 | -0.05 | 0.1 | -0.21 | 0.21 | -0.12 | 0.19 |
| | | 10% | 9 | 0.05 | -0.1 | 0.21 | -0.21 | 0.12 | -0.19 |
| | | 20% | 9 | 0.1 | -0.2 | 0.42 | -0.42 | 0.24 | -0.37 |
| $I_c$ | 0.05 | -20% | 9 | -0.1 | 0.21 | -0.42 | 0.43 | -0.24 | 0.38 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | -10% | 9 | -0.05 | 0.1 | -0.21 | 0.21 | -0.12 | 0.19 |
| | | 10% | 9 | 0.05 | -0.1 | 0.21 | -0.21 | 0.12 | -0.19 |
| | | 20% | 9 | 0.1 | -0.2 | 0.42 | -0.42 | 0.24 | -0.37 |
| $\alpha$ | 0.4 | -20% | 9 | -0.1 | 0.21 | -0.42 | 0.43 | -0.24 | 0.38 |
| | | -10% | 9 | -0.05 | 0.1 | -0.21 | 0.21 | -0.12 | 0.19 |
| | | 10% | 9 | 0.05 | -0.1 | 0.21 | -0.21 | 0.12 | -0.19 |
| | | 20% | 9 | 0.1 | -0.2 | 0.42 | -0.42 | 0.24 | -0.37 |
| $\gamma$ | 0.1 | -20% | 6 | -0.2 | -2.9 | -6.55 | -10.76 | -12.4 | -5.08 |
| | | -10% | 7 | -0.14 | -2.04 | -4.66 | -6.82 | -8.04 | -2.68 |
| | | 10% | 10 | 0.04 | 0.52 | 1.24 | 4.13 | 4.49 | 2.97 |
| | | 20% | 12 | 0.14 | 2.07 | 5 | 11.09 | 12.62 | 6.23 |
| $G$ | 50 | -20% | 11 | -0.06 | -0.89 | -2.06 | 0.63 | 0.05 | 2.56 |
| | | -10% | 10 | -0.02 | -0.3 | -0.69 | 0.56 | 0.36 | 1.2 |
| | | 10% | 8 | 0.0004 | 0.01 | 0.01 | -1.04 | -1.04 | -1.05 |
| | | 20% | 7 | 0.02 | 0.28 | -0.66 | -2.58 | -2.76 | -1.98 |

From Table 2, the following observations are made:

(i) The optimal values of selling price (*p*), stock in period without shortage case $(t_1)$, time period of shortage $(t_2)$, initial inventory level (*S*), maximum shortage (*R*)and total profit per unit time are insensitive with respect to the location parameter of holding cost $(g)$, shape parameter of holding cost $(h)$, shortage cost $(C_s)$, lost sale cost $(C_l)$, lead time $(L)$, rate of chargeable interest $(I_c)$ ,partial payment fraction $(\alpha)$ and advertisement cost $(G)$. However, the initial highest stock level is less sensitive with respect to the location parameter of holding cost $(g)$. On the other hand, highest shortage level is less sensitive with respect to shortage cost $(C_s)$ and opportunity cost $(C_l)$.

(ii) The total profit per unit time is highly sensitive with respect to the location parameter $a$ and $b$. The location parameter of the demand ( $a$ ) has the greatest positive impact on increasing the total profit. Notice that a higher value of $a$ helps to augment the customers' demand and consequently, to increase the retailer's total revenue. On the other hand, the shape parameter of the demand rate $b$ has the greatest negative impact on the total profit. Note that higher value of $b$ causes to abate the customers' demand and so, to abate the retailer's total revenue. So this suggests to the decision maker (retailer) must give foremost attention on the demand parameters instead of reducing all costs for increasing the total profit.

(iii) Maximum shortage level (*R*) is highly sensitive with respect to the changes in replenishment cost $(C_o)$, advertisement elasticity $(\gamma)$, backlogging rate $(\delta)$ and location parameter $(a)$. Among the mentioned parameters, the backlogging parameter $(\delta)$ has the greatest impact on *R*; decreasing it.

(iv) Highest stock level ($S$) is highly sensitive in a positive way with respect to the changes in location parameter ($a$), advertisement elasticity $(\gamma)$ and maximum lifetime ($E$). This shows the fact that if $a$ increases, then the customers' demand also increases and so, the retailer need to store a large amount of products to handle customers' high demand. For a higher maximum lifetime product, the retailer orders a large amount of products in order to reduce ordering cost. On the other hand, $S$ is highly sensitive in a negative way with respect to the changes in shape parameter of demand ($b$)and purchasing cost ($C_p$). For a higher value of $b$, customers' demand decreases significantly and then, retailer needs to store a small amount of products in order to avoid the higher carrying cost. For a higher unit purchase cost, the retailer orders a small amount of the product.

(v) The parameters replenishment cost ($C_0$) shape parameter of demand ($b$), purchasing cost ($C_p$), holding cost parameters $(g \text{ and } h)$, interest charge rate ($I_c$), fraction amount of prepayment on purchasing cost ($\alpha$), advertisement elasticity $(\gamma)$ and advertisement cost ($G$) have an impact on the optimal shortages period $(t_2)$ in positive way. For example, if the values of the mentioned parameters increase, then the optimal shortages period also increases. Among these parameters, the shape parameter $(b)$ of the demand has the greatest impact on $t_2$. This reveals that if $b$ increases, then customers' demand decreases and hence, the retailer orders a small amount of products. Consequently, all products are consumed within a short period and shortages period augments significantly. On the other hand, the parameters location parameter ($a$), purchasing cost ($C_p$), shortage cost ($C_s$), opportunity cost ($C_l$), backlogging rate $(\delta)$, maximum lifetime ($E$) and number of equal pre-payment ($n$) have impact on $t_2$ as follows if the values of the mentioned parameters increase, then $t_2$ decreases. The location parameter $(a)$ of the demand has the greatest effect on $t_2$; for example, a higher value of $a$ increases the demand significantly and hence, the retailer makes a large order size and the shortages reduces significantly.

(vi) Price of the product $(p)$ is highly sensitive with respect to the demand parameter $a$ in a positive way and it is also highly sensitive with respect to the other demand parameter $b$ but in a negative way. The price $p$ is moderately sensitive relating to the unit purchase cost whereas less sensitive with regard to the rest of the parameters. Another interesting observation is that a higher selling price does not always give a higher total profit.

(vii) The stock-in period $(t_1^*)$ is highly sensitive with respect to maximum lifetime $(E)$ and unit purchase cost $(C_p)$. It reveals that, on the one hand, for a higher maximum lifetime product the retailer (decision maker) orders a large amount of products. On the other hand, for a higher unit purchase cost retailer orders a small amount of product. Also $(t_1^*)$ is moderately sensitive with respect to the parameters replenishment cost $(C_o)$, location parameter $(a)$, shape parameter $(b)$, fixed parameter of the holding cost $(g)$ and advertisement elasticity $(\gamma)$. However, the scale parameter of the holding cost $(h)$, shortage cost $(C_s)$, lost sale cost $(C_l)$, rate of chargeable interest $(I_c)$, backlogging rate $(\delta)$, partial payment fraction $(\alpha)$, number of equal pre-payment $(n)$ and advertisement cost $(G)$ have almost no impact on $t_1$.

(viii) The frequency of advertisement $(A)$ is highly sensitive with respect to the parameters purchasing cost $(C_p)$, location parameter $(a)$, shape parameter $(b)$, advertisement elasticity $(\gamma)$, maximum lifetime $(E)$ and advertisement cost $(G)$. Among these parameters, the location parameter of demand $(a)$, maximum life time $(E)$ and advertising elasticity $(\gamma)$ have impact on $A$ as follows if the values of the mentioned parameters increase, then $A$ also increases. On the other hand, shape parameter $(b)$, purchasing cost $(C_p)$ and advertisement cost $(G)$ affect the optimal frequency of advertisement in a negative way.

## 7. Managerial implications

The following findings are obtained from the sensitivity analysis. These are also the suggestions to the manager/ the decision maker in order to enhance the total profit of the organization.

- As the location parameter $(a)$ and shape parameter $(b)$ of the demand rate have the greatest effect on the retailer's total profit in positive sense and negative manner respectively, the decision maker must give a meticulous concentration on boosting the customers' demand by implementing an effective marketing policy instead of reducing inventory related costs.
- As a much effective advertisement boosts the customers' demand in a great manner by creating the brand awareness and promoting the information of the products to the potential customers, therefore manager must develop an effective advertisement and telecast through a popular media with affordable cost per advertisement by negotiating with the media companies.
- The sensitivity analysis reveals that the total profit augments when the number of equally spaced prepayments increases during the lead time. For that reason, the manager must select the manufacturer or supplier who allows a higher number of equally spaced advance payment policies with a small portion of the total purchasing cost.
- The deterioration rate is an increasing function of holding time of the product and this rate increases as much as the product approaches to their maximum lifetime or expired date. For this reason, the customers always want to purchase the products which have longer expiration dates so

that the products can be preserved for a longer time. Consequently, the products with a higher maximum lifetime attract more customers and help to augment the revenue in a greater extent. Hence, the manager is suggested to select the products with higher maximum lifetimes.

- The sensitivity analysis also shows that the unit purchase cost has the greatest impact on the retailer's total profit per unit time among all the cost parameters. Thus, the manager must reduce the unit purchase cost by negotiating with the suppliers/manufacturers ensuring them that they are going to make a higher order size if the unit purchase cost is lower.
- The retailers always try to run their business without any discontinuation not only to keep their popularity in the today's competitive market environment but also to hold their regular customers meeting the demands. Furthermore, inventory systems are not always profitable. When the inventory system is not profitable, the decision maker should avoid the advertisement policy.

## 8. Conclusions

This research work develops two inventory models (without shortages and with shortages) considering advanced payment for a deteriorating product which has a maximum lifetime. The inventory models assume that the holding cost follows a linearly time-dependent increasing function. The demand is dependent on the selling price as well as the frequency of advertisement when advertisement number is confined to be a positive integer. In both inventory models, a mixed-integer constraint optimization is formulated and solved by exploring the existence of the optimal values of the decision variables for maximizing the total profit. To validate proposed inventory models, three numerical examples are presented and solved by dint of the proposed solution algorithms. In the bottom line, several managerial insights are found by performing a sensitivity analysis observing the effect of changes on different parameters. When the inventory system is not profitable, the decision maker should avoid the advertisement policy. The retailer should select the supplier who provides the products with higher maximum lifetimes of the particular type of item and also allows a higher number of equally spaced advance payment policies with a small portion of the total purchasing cost.

For further research, one can extend the proposed inventory model with shortages by including realistic features such as nonlinear demand, nonlinear holding cost, stock-dependent demand, stock-dependent demand, and time-dependent demand under inflation. Moreover, allowing multiple delayed payments within equal sized credit periods from the delivery time and quantity discounts based on the order size can be salient extensions. Also, the model inventory model with shortages can be explored by introducing imprecise environments such as fuzzy-valued inventory parameters and interval-valued inventory parameters. These are some interesting research directions that the researchers and academicians can do in the near future.

**Appendix A. Proof of the Theorem 1.**

For convenience, denote

$$f_1(T)=(A+1)^{\gamma}(a-bp)XT-\left(\begin{array}{l}C_0+GA+Y(A+1)^{\gamma}(a-bp)(1+E)\ln\left(\dfrac{1+E}{1+E-T}\right)\\+g(A+1)^{\gamma}(a-bp)\dfrac{T^2}{4}+h(A+1)^{\gamma}(a-bp)\left\{\dfrac{T^3}{9}-\dfrac{(1+E)T^2}{12}\right\}\end{array}\right),$$

and

$g_1(T)=T$.

Now Eq. (4) is written in the form: $\Pi_1(A,p,T)=\dfrac{f_1(T)}{g_1(T)}$. The first and second order derivatives of $f_1(T)$ for a fixed $p$ with respect to $T$ are given by:

$$f_1'(T)=(A+1)^{\gamma}(a-bp)\left[X-\left(Y\frac{1+E}{1+E-T}+g\frac{T}{2}+h\left\{\frac{T^2}{3}-\frac{(1+E)T}{6}\right\}\right)\right],$$

and

$$f_1''(T)=-(A+1)^{\gamma}(a-bp)\left[\left\{\left(1+\frac{n+1}{2n}I_cLK\right)C_p+\frac{g}{2}(1+E)\right\}\frac{1+E}{(1+E-T)^2}+\frac{1}{2}g+\frac{1}{6}h\left\{\frac{(1+E)^3}{(1+E-T)^2}+4T-(1+E)\right\}\right].$$

If the value of the expression $\dfrac{(1+E)^3}{(1+E-T)^2}+4T-(1+E)$ is always non-negative then $f_1(T)$ is strictly concave. Now the expression $\dfrac{(1+E)^3}{(1+E-T)^2}+4T-(1+E)$ is always positive for any $T>0$ as $\dfrac{(1+E)^3}{(1+E-T)^2}-(1+E)>0$. For that reason, $f_1''(T)$ is a negative valued function for all $T>0$ and hence, the $f_1(T)$ is a differentiable and concave function. Moreover, $g_1(T)=T$ is a positive and affinefunction. For a given value of $A$ and $p$; $\Pi_1(A,p,T)$ is a strictly pseudo-concave function of $T$ and therefore, there exists a unique optimal solution $T^*$. Consequently, the part (a) of the Theorem 1 is proved.

The first order partial derivative of $\Pi_1(A,p,T)$ with respect to $T$ is:

$$\frac{\partial \Pi_1(A,p,T)}{\partial T} = (A+1)^{\gamma}(a-bp)\left[\begin{array}{l}\frac{(C_0+GA)}{(A+1)^{\gamma}(a-bp)T^2} + Y(1+E)\left\{\frac{1}{T^2}\ln\left|\frac{1+E}{1+E-T}\right| - \frac{1}{T(1+E-T)}\right\} \\ -\frac{g}{4} - h\left\{\frac{2T}{9} - \frac{(1+E)}{12}\right\}\end{array}\right]. \qquad \text{(A1)}$$

Note that it is not possible to find the value of $\frac{\partial \Pi_1(A,p,T)}{\partial T}$ at the replenishment cycle length $T=0$ but the limiting value at $T=0$ with the help of L'Hopital's rule can be easily determined. For the convenience, first note that

$$\lim_{T\to 0}\left\{\frac{1}{T^2}\ln\left|\frac{1+E}{1+E-T}\right| - \frac{1}{T(1+E-T)}\right\} = \lim_{T\to 0}\left[\frac{(1+E-T)\ln\left|\frac{1+E}{1+E-T}\right| - T}{T^2(1+E-T)}\right]$$

$$= \lim_{T\to 0}\left[\frac{-\ln\left|\frac{1+E}{1+E-T}\right|}{2T(1+E)-3T^2}\right] = \lim_{T\to 0}\left[\frac{-\frac{1}{1+E-T}}{2(1+E)-6T}\right] = -\frac{1}{2(1+E)^2}.$$

Thus, from Eq. (A1) ,

$$\lim_{T\to 0}\frac{\partial \Pi_1(A,p,T)}{\partial T} = \infty. \qquad \text{(A2)}$$

The value of $\frac{\partial \Pi_1(A,p,T)}{\partial T}$ at $T=E$ is given by:

$$\left.\frac{\partial \Pi_1(A,p,T)}{\partial T}\right|_{T=E} = (A+1)^{\gamma}(a-bp)\left[\frac{(C_0+GA)}{(A+1)^{\gamma}(a-bp)E^2} + Y\frac{(1+E)}{E^2}\{\ln|1+E| - E\} - \frac{g}{4} - \frac{h}{36}(5E-3)\right] = \Delta_1.$$

On the one hand, if $\Delta_1 > 0$ then $\Pi_1(A,p,T)$ is a strictly increasing function over the entire cycle length $[0,T]$ where $0 < T \le E$. Consequently, the total profit per unit time is maximized at $T^* = E$. With this, the proof of part (b) of Theorem 1 is completed.

On the other hand, if $\Delta_1 \le 0$ then there exists a unique point between 0 and $E$ where the profit function $\Pi_1(A,p,T)$ attains its global maximum value. This is the proof of part (c) of Theorem 1.

**Appendix B. Proof of the Theorem 2.**

The first order and second order partial derivatives of $\Pi_1(A,p,T)$ with respect to $p$ are:

$$\frac{\partial \Pi_1(A,p,T)}{\partial p}=\frac{(A+1)^{\gamma}}{T}\left[-bXT+(a-bp)T+b\left(Y(1+E)\ln\left|\frac{1+E}{1+E-T}\right|+g\frac{T^2}{4}+h\left\{\frac{T^3}{9}-\frac{(1+E)T^2}{12}\right\}\right)\right], \quad \text{(B1)}$$

and

$$\frac{\partial^2 \Pi_1(A,p,T)}{\partial p^2}=-2b(A+1)^{\gamma}<0. \quad \text{(B2)}$$

Consequently, $\Pi_1(A,p,T)$ is a concave function with respect to $p$ for a given value of $A>0$ and $T>0$ within $[0,\infty)$ and thus there exists a unique optimal solution $p^* \in [0,\infty)$ satisfying $\frac{\partial \Pi_1(\cdot)}{\partial p}=0$ such that profit function per unit time is maximized. After rearranging the terms, Eq. (B1) is rewritten as:

$$\frac{\partial \Pi_1(\cdot)}{\partial p}=(a-2bp)(A+1)^{\gamma}+\frac{(A+1)^{\gamma}b}{T}\left[\begin{array}{l} C_p(1+E)\ln\left|\frac{1+E}{1+E-T}\right|+\frac{n+1}{2n}I_c\alpha LC_p(1+E)\ln\left|\frac{1+E}{1+E-T}\right| \\ +\left\{\frac{g}{2}(1+E)+\frac{h}{6}(1+E)^2\right\}(1+E)\ln\left|\frac{1+E}{1+E-T}\right| \\ +g\left\{\frac{T^2}{4}-\frac{(1+E)T}{2}\right\}+h\left\{\frac{T^3}{9}-\frac{(1+E)T^2}{12}-\frac{(1+E)^2T}{6}\right\} \end{array}\right]. \quad \text{(B3)}$$

In the right hand side of Eq. (B3), specifically inside of brackets the first term is a positive factor of purchasing cost (*PC*), the second term is a positive factor of interest of loan (*IL*) and the third, fourth and fifth terms are positive factors of holding cost (*HC*). As a result, $\frac{\partial \Pi_1(\cdot)}{\partial p}=0$ is solvable for $p$ if $a-2bp<0$ i.e., $p>\frac{a}{2b}$. But to ensure the demand $D(A,p)\geq 0$, the selling price per unit $p$ must be equal or less than $\frac{a}{b}$ i.e., $p\leq\frac{a}{b}$. Combining these two inequalities, it is concluded that there exists a unique $p^*\in\left(\frac{a}{2b},\frac{a}{b}\right]$ such that profit function per unit time is maximized. On the other hand, if $\frac{\partial \Pi_1(\cdot)}{\partial p}=0$ provides the unit selling price $p>\frac{a}{b}$, then the customers' demand rate ($(A+1)^{\gamma}(a-bp)$) becomes negative which contradicts the practical scenario. In this case, the optimal selling price is $p^*=\frac{a}{b}$ .This completes the proof.

**Appendix C.**

For any given $p, T > 0$, the first order partial derivative of $\Pi_1(A, p, T)$ with respect to $A$ is:

$$\frac{\partial \Pi_1(\cdot)}{\partial A} = \frac{1}{T}\left[\begin{array}{l} \gamma(A+1)^{\gamma-1}(a-bp)XT - G \\ -\gamma(A+1)^{\gamma-1}(a-bp)\left(Y(1+E)\ln\left|\frac{1+E}{1+E-T}\right| + g\frac{T^2}{4} + h\left\{\frac{T^3}{9} - \frac{(1+E)T^2}{12}\right\}\right) \end{array}\right]. \tag{C1}$$

Again, the second order partial derivative of $\Pi_1(A, p, T)$ with respect to $A$ is:

$$\frac{\partial^2 \Pi_1(\cdot)}{\partial A^2} = \frac{\gamma(\gamma-1)(A+1)^{\gamma-2}(a-bp)}{T}\left[\begin{array}{l} XT \\ -\left(Y(1+E)\ln\left|\frac{1+E}{1+E-T}\right| + g\frac{T^2}{4} + h\left\{\frac{T^3}{9} - \frac{(1+E)T^2}{12}\right\}\right) \end{array}\right]. \tag{C2}$$

After rearranging the terms of (C2), one has

$$\frac{\partial^2 \Pi_1(\cdot)}{\partial A^2} = \frac{\gamma(\gamma-1)}{T(A+1)^2}\left[\begin{array}{l} p(A+1)^{\gamma}(a-bp)T - \left(1+\frac{n+1}{2n}I_c\alpha L\right)C_p\left\{(A+1)^{\gamma}(a-bp)(1+E)\ln\left|\frac{1+E}{1+E-T}\right|\right\} \\ -\left\{\frac{g}{2}(1+E)+\frac{h}{6}(1+E)^2\right\}(A+1)^{\gamma}(a-bp)(1+E)\ln\left|\frac{1+E}{1+E-T}\right| \\ -g(A+1)^{\gamma}(a-bp)\left\{\frac{T^2}{4}-\frac{(1+E)T}{2}\right\} - h(A+1)^{\gamma}(a-bp)\left\{\frac{T^3}{9}-\frac{(1+E)T^2}{12}-\frac{(1+E)^2 T}{6}\right\} \end{array}\right]. \tag{C3}$$

In the right hand side of (C3), specifically inside of the third bracket the first term is the total revenue and remaining all terms are different inventory costs during the entire cycle length. When the summation of all terms inside the third bracket is positive, the model is profitable and then $\frac{\partial^2 \Pi_1(\cdot)}{\partial A^2} < 0$. Thus, the total profit per unit time is concave with respect to $A$ in $[0, \infty)$. On the other hand, when the model is not profitable, then $XT - \left(Y(1+E)\ln\left|\frac{1+E}{1+E-T}\right| + g\frac{T^2}{4} + h\left\{\frac{T^3}{9} - \frac{(1+E)T^2}{12}\right\}\right) \le 0$. In this situation, from Eq. (C1), $\frac{\partial \Pi_1(\cdot)}{\partial A} < 0$ which reveals that the total profit function per unit time is always decreasing function for $A$ in $[0, \infty)$. So, the optimal total profit function per unit time can be found at $A = 0$.

**Appendix D. Proof of the Theorem 3.**

For convenience, define the following auxiliary functions from Eq. (12)

$$f_2(t_1,t_2)=(A+1)^{\gamma}(a-bp)\begin{bmatrix} Xt_1-\dfrac{C_0+GA}{(A+1)^{\gamma}(a-bp)}-Y(1+E)\ln\left|\dfrac{1+E}{1+E-t_1}\right|-g\dfrac{t_1^{\,2}}{4} \\ -h\left[\dfrac{t_1^{\,3}}{9}-\dfrac{(1+E)t_1^{\,2}}{12}\right]+X_1\dfrac{\ln|1+\delta t_2|}{\delta}-\left(C_l+\dfrac{C_s}{\delta}\right)t_2 \end{bmatrix}, \tag{D1}$$

and

$$g_2(t_1,t_2)=t_1+t_2. \tag{D2}$$

Therefore, from Eq. (12), the retailer's total profit per unit time, for any given positive values of $A$ and $p$, becomes $\Pi_2(A,p,t_1,t_2)=\dfrac{f_2(t_1,t_2)}{g_2(t_1,t_2)}$. In order to prove that the $\Pi_2(A,p,t_1,t_2)$ is jointly pseudo-concave function in $(t_1,t_2)$ it is necessary to show that $f_2(t_1,t_2)$ is a differentiable and (strictly) joint concave function in $(t_1,t_2)$. To construct the Hessian matrix for $f_2(t_1,t_2)$, calculate all possible second order partial derivatives with respect to $t_1$ and $t_2$.

$$\frac{\partial^2 f_2(\cdot)}{\partial t_1^{\,2}}=-(A+1)^{\gamma}(a-bp)\left[Y\frac{1+E}{(1+E-t_1)^2}+\frac{g}{2}+h\left\{\frac{2t_1}{3}-\frac{(1+E)}{6}\right\}\right]$$

$$=-(A+1)^{\gamma}(a-bp)\begin{bmatrix}\left(1+\dfrac{n+1}{2n}I_c\alpha L\right)C_p\dfrac{1+E}{(1+E-t_1)^2} \\ +\dfrac{g}{2}\left\{\dfrac{(1+E)^2}{(1+E-t_1)^2}+1\right\}+\dfrac{h}{6}\left\{\dfrac{(1+E)^3}{(1+E-t_1)^2}+4t_1-(1+E)\right\}\end{bmatrix}, \tag{D3}$$

$$\frac{\partial^2 f_2(\cdot)}{\partial t_1\partial t_2}=\frac{\partial^2 f_2(\cdot)}{\partial t_2\partial t_1}=0, \tag{D4}$$

and

$$\frac{\partial^2 f_2(\cdot)}{\partial t_2^{\,2}}=-(A+1)^{\gamma}(a-bp)X_1\frac{\delta}{(1+\delta t_2)^2}. \tag{D5}$$

Then, the Hessian matrix for $f_2(t_1,t_2)$ is:

$$H_{ii}=\begin{bmatrix}\dfrac{\partial^2 f_2(\cdot)}{\partial t_1^{\,2}} & \dfrac{\partial^2 f_2(\cdot)}{\partial t_1\partial t_2} \\ \dfrac{\partial^2 f_2(\cdot)}{\partial t_2\partial t_1} & \dfrac{\partial^2 f_2(\cdot)}{\partial t_2^{\,2}}\end{bmatrix}.$$

The first principal minor is

$$\left|H_{11}\right| = \frac{\partial^2 f_2(\cdot)}{\partial t_1{}^2}$$

$$= -(A+1)^{\gamma}(a-bp)\left\{\left(1+\frac{n+1}{2n}I_c\alpha L\right)C_p\,\frac{1+E}{(1+E-t_1)^2}+\frac{g}{2}\left[\frac{(1+E)^2}{(1+E-t_1)^2}+1\right]+\frac{h}{6}\left[\frac{(1+E)^3}{(1+E-t_1)^2}+4t_1-(1+E)\right]\right\}.$$

Notice that the expression $\frac{(1+\tau)^3}{(1+\tau-t_1)^2}+4t_1-(1+E)$ is always positive for any $t_1>0$ and as a result, $\left|H_{11}\right|<0$.

Again, the second principal minor is

$$\left|H_{22}\right| = \frac{\partial^2 f_2(\cdot)}{\partial t_1{}^2}\frac{\partial^2 f_2(\cdot)}{\partial t_2{}^2}-\frac{\partial^2 f_2(\cdot)}{\partial t_2\partial t_1}\frac{\partial^2 f_2(\cdot)}{\partial t_1\partial t_2}$$

$$=(A+1)^{2\gamma}(a-bp)^2\,\frac{X_1\delta}{(1+\delta t_2)^2}\left\{\begin{aligned}&\left(1+\frac{n+1}{2n}I_c\alpha L\right)C_p\,\frac{1+E}{(1+E-t_1)^2}+\frac{g}{2}\left[\frac{(1+E)^2}{(1+E-t_1)^2}+1\right]\\&+\frac{h}{6}\left[\frac{(1+E)^3}{(1+E-t_1)^2}+4t_1-(1+E)\right]\end{aligned}\right\}>0.$$

As the first principal minor is negative and the second principal minor is positive, the Hessian matrix for $f_2(t_1,t_2)$ is negative definite. For that reason, $f_2(t_1,t_2)$ is a differentiable and (strictly) concave function with respect to $t_1$ and $t_2$ simultaneously. Moreover, the function $g_2(t_1,t_2)=t_1+t_2$ is a positive, differentiable and affine function, so the retailer's total profit function per unit time $\Pi_2(A,p,t_1,t_2)$ is jointly pseudo-concave function in $(t_1,t_2)$, and has only one maximum value.

As a result, the objective function $\Pi_2(A,p,t_1,t_2)$, for any fixed $A$ and $p$, attains the global maximum value for a unique pair of values $(t_1,t_2)$. This completes the proof.

**Appendix E. Proof of the Theorem 4.**

Performing the implicit differentiation in Eq. (25) with respect to $t_1$, one has

$$\frac{\delta X_1}{1+\delta t_2}\frac{dt_2}{dt_1}=Y\,\frac{1+E}{(1+E-t_1)^2}+\frac{g}{2}+h\left(\frac{2t_1^2}{3}-\frac{1+E}{6}\right)>0. \tag{E1}$$

The above result reveals that, for any fixed $A$ and $p$, the stock-out period $t_2$ is an increasing function of stock-in period $t_1$.

The first order partial derivative of $\Pi_2(A,p,t_1,t_2)$ with respect to $t_1$ by dint of Eq. (25) can be expressed as follows:

$$\frac{\partial \Pi_2(\cdot)}{\partial t_1} = \frac{(A+1)^{\gamma}(a-bp)}{t_1+t_2}\left\{\begin{array}{l} \dfrac{C_0+GA}{(A+1)^{\gamma}(a-bp)} + Y(1+E)\ln\left|\dfrac{1+E}{1+E-t_1}\right| + g\dfrac{t_1^{\,2}}{4} \\ +h\left[\dfrac{t_1^{\,3}}{9} - \dfrac{(1+E)t_1^{\,2}}{12}\right] - X_1\dfrac{\ln\left|1+\delta t_2\right|}{\delta} + X_1\dfrac{t_1+t_2}{1+\delta t_2} - \left(X + C_l + \dfrac{C_s}{\delta}\right)t_1 \end{array}\right\} \text{.(E2)}$$

If $t_1 \to 0$, then one can find also $t_2 \to 0$ from Eq. (25). It is not possible to find the value of $\frac{\partial \Pi_2(\cdot)}{\partial t_1}$ at $t_1 = 0$ but the limiting value at $t_1 = 0$ with the help of L'Hospital's rule can be easily determined. Then, for any given value of $A \geq 0$ and $p > 0$, one can easily find that $\lim_{t_1 \to 0} \frac{\partial \Pi_2(\cdot)}{\partial t_1} = \infty$.

Again, the value of $\frac{\partial \Pi_2(\cdot)}{\partial t_1}$ at $t_1 = E$

$$\left.\frac{\partial \Pi_2(\cdot)}{\partial t_1}\right|_{t_1=E} = \frac{(A+1)^{\gamma}(a-bp)}{E+\frac{1}{\delta}\{F(E)-1\}}\left\{\begin{array}{l} \dfrac{C_0+GA}{(A+1)^{\gamma}(a-bp)} + Y(1+E)\ln\left|1+E\right| + g\dfrac{E^2}{4} + h\left[\dfrac{E^3}{9} - \dfrac{(1+E)E^2}{12}\right] \\ -X_1\dfrac{\ln\left|F(t_1)\right|}{\delta} + X_1\dfrac{\delta E - 1 + F(t_1)}{\delta F(t_1)} - \left(X + C_l + \dfrac{C_s}{\delta}\right)E. \end{array}\right\} \text{.(E3)}$$

In addition, according to the result of Theorem 4, $\Pi_2(A,p,t_1,t_2)$ is a pseudo-concave function of $t_1$ and $t_2$.

If $\Delta_2 > 0$, then $\Pi_2(A,p,t_1,t_2)$ is a strictly increasing function with respect to $t_1$ over the interval $[0, E]$. Consequently, total profit is maximized at $t_1^{*} = E$ and the corresponding optimal value of $t_2^{*}$, from Eq. (25), is $\frac{1}{\delta}\{F(E)-1\}$. Hence, the proof of part (a) of Theorem 4 is completed.

On the other hand, if $\Delta_2 \leq 0$, then there exists a unique value of $t_1$ between 0 and $E$ where the profit function $\Pi_2(A,p,t_1,t_2)$ attains its global maximum value (by mean value theorem). This is the proof of part (b) of Theorem 4. The corresponding optimal value of $t_2^{*}$ can be obtained from Eq. (25).

**Appendix F. Proof of the Theorem 5.**

For a given value of $A \geq 0, t_1 > 0$ and $t_2 > 0$, the first order and second order partial derivatives of $\Pi_2(A,p,t_1,t_2)$ with respect to $p$ are:

$$\frac{\partial \Pi_2(\cdot)}{\partial p} = \frac{(A+1)^{\gamma}}{t_1 + t_2}\left[\begin{array}{l}(a-bp)t_1 - bXt_1 + bY(1+E)\ln\left|\frac{1+E}{1+E-t_1}\right| + gb\frac{t_1^{\ 2}}{4} + hb\left[\frac{t_1^{\ 3}}{9} - \frac{(1+E)t_1^{\ 2}}{12}\right] \\ +(a-bp)\frac{\ln\left|1+\delta t_2\right|}{\delta} - bX_1\frac{\ln\left|1+\delta t_2\right|}{\delta} + \left(C_l + \frac{C_s}{\delta}\right)bt_2\end{array}\right], \tag{F1}$$

and

$$\frac{\partial^2 \Pi_2(\cdot)}{\partial p^2} = -\frac{2(A+1)^{\gamma} b}{t_1 + t_2}\left[t_1 + \frac{\ln\left|1+\delta t_2\right|}{\delta}\right] < 0. \tag{F2}$$

Thus, $\Pi_2(A, p, t_1, t_2)$ is a concave function in $p$ for a given value of $A > 0, t_1 > 0$ and $t_2 > 0$ within $[0, \infty)$ and for that reason, there exists a unique optimal solution $p^* \in [0, \infty)$ satisfying $\frac{\partial \Pi_2(\cdot)}{\partial p} = 0$ such that $\Pi_2(A, p, t_1, t_2)$ is maximized. After rearranging the terms, Eq. (F1) is expressed as follows:

$$\begin{aligned}\frac{\partial \Pi_2(\cdot)}{\partial p} &= (a-2bp)\frac{(A+1)^{\gamma}}{t_1+t_2}\left[t_1 + \frac{\ln\left|1+\delta t_2\right|}{\delta}\right] \\ &+ \frac{b(A+1)^{\gamma}}{t_1+t_2}\left\{\begin{array}{l} g\left[\frac{(1+E)^2}{2}\ln\left|\frac{1+E}{1+E-t_1}\right| + \frac{t_1^{\ 2}}{4} - \frac{g(1+E)t_1}{2}\right] + \\ h\left[\frac{(1+E)^3}{6}\ln\left|\frac{1+E}{1+E-t_1}\right| + \frac{t_1^{\ 3}}{9} - \frac{(1+E)t_1^{\ 2}}{12} - \frac{h(1+E)^2 t_1}{6}\right] + C_p\left[(1+E)\ln\left|\frac{1+E}{1+E-t_1}\right| + \frac{\ln\left|1+\delta t_2\right|}{\delta}\right] \\ + \frac{n+1}{2n} I_c \alpha L C_p\left[(1+E)\ln\left|\frac{1+E}{1+E-t_1}\right| + \frac{\ln\left|1+\delta t_2\right|}{\delta}\right] + \left(C_l + \frac{C_s}{\delta}\right)b\left(t_2 - \frac{\ln\left|1+\delta t_2\right|}{\delta}\right)\end{array}\right\}.\end{aligned} \tag{F3}$$

Every term inside the brackets of the right hand side of Eq. (F3) is positive because of these are positive factors of different inventory costs. Consequently, $\frac{\partial \Pi_2(\cdot)}{\partial p} = 0$ is solvable for $p$ if $a - 2bp < 0$ i.e., $p > \frac{a}{2b}$. But to ensure the demand $D(A, p) \geq 0$, the selling price per unit $p$ must be equal or less than $\frac{a}{b}$ i.e., $p \leq \frac{a}{b}$.As a result, combining these two inequalities, there exists a unique $p^* \in \left(\frac{a}{2b}, \frac{a}{b}\right]$ such that the profit function per unit time is maximized. On the other hand, if $\frac{\partial \Pi_2(\cdot)}{\partial p} = 0$ provides the unit selling price $p > \frac{a}{b}$, then the customers' demand rate ( $(A+1)^{\gamma}(a-bp)$ ) becomes negative which contradicts the practical scenario. In this case, the optimal selling price is $p^* = \frac{a}{b}$. This completes the proof.


**Acknowledgments**

The authors thank the three anonymous referees for their constructive comments and suggestions that improved the paper.


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