The effect of advance payment with discount facility on supply decisions of deteriorating products whose demand is both price and stock dependent

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Abstract

Generally, in the business world, it is observed that suppliers give different kinds of benefits to retailers due to advance payment. One of the popular benefits is instant cash discount due to advance payment. If a retailer pays off his total purchase cost before receiving the products, then he receives a certain percentage of cash discount instantly. However, if the retailer pays off a certain fraction of the total purchasing cost, then price discount is given only at the time of receiving the products while paying the rest amount of the total purchasing cost. Using this concept, this paper formulates, under both cases of advance payment (full or partial), an inventory model for deteriorating products where shortages are allowed and demand function is considered as price and stock-dependent. The closed-form solutions for each case are presented and two numerical examples are solved. In addition, a sensitivity analysis is also performed to show the effects of advance payment with discount facility.

Keywords: Pricing; deterioration; advance payment; discount; price and stock-dependent demand

1. Introduction

In the present day competitive market dynamics, it is very challenging for businessmen and sellers to attract and convert potential customers into actual buyers. In this end, to catch the attention of the customers, vendors embark on different penetrations and marketing strategies as market conditions never remain stable for long periods. In order to run their business successfully, vendors offer different types of discounts, namely, quantity discounts, trade discounts, promotional discounts, seasonal discounts, cash discounts and pre-payment facilities, etc. Among these, quantity and trade discounts are most common in practice. In recent times, provisions for pre-payment facilities have been in vogue and are offered by vendors to attract their potential buyers in prevailing market situations. Our proposed model is applicable in every sector in related with the deteriorating items such as shopping mall, wholesale market, retail business, online shopping, etc.

At present, the concept of pre-payment has caught the attention of a lot of researchers and academicians. Recently, Taleizadeh (2014a) presented an economic order quantity model by assuming that the vendor requests to his buyer to prepay a fraction of the purchasing cost in equal-sized in multiple installments. The remaining payment is made at the time of receiving the lot. Taleizadeh's (2014a) model considers deteriorating items under constant demand where shortages are completely backlogged. In another paper, Taleizadeh (2014b) improved his previous work assuming that shortages are partially backlogged. But in reality, the buyer most of the time wants to avoid this type of situation because, in the case of prepayment, he has to bear some additional cost on the amount that has been prepaid. In such cases, to create a center of attention of buyers, vendors can offer a discount based on the amount of prepayment. Also in both of the aforementioned studies (Taleizadeh2014a, Taleizadeh2014b), demand rate of the product is considered as constant, although, in practice, it is influenced by the selling price as well as the stock level of the product.

In recent years, inventory models for deteriorating items have widely investigated by eminent researchers/academicians. For instance, Chung and Cárdenas-Barrón (2013) analyzed an inventory model for a deteriorating product under stock-dependent demand and two-level trade credit system. Yang et al. (2013) introduced retailer's optimal ordering policy when suppliers offer a cash discount or delay in payment facility. Sarkar and Sarkar (2013) developed an economic order quantity inventory model for infinite replenishment rate with stock-dependent demand and time-varying deterioration. Shah and Cárdenas-Barrón (2015) presented a retailer's credit policy for deteriorating items with order linked credit period and cash discount. Most recently, Teng et al. (2016) studied an inventory model for deteriorating items with seasonal demand where deterioration rate gradually increases as the expiration date approaches. Wu et al. (2016) formulated an inventory model with maximum lifetime of the product under downstream partial trade credit and credit risk customers. Khan et al. (2019) considered an inventory model in all units discount environment for a maximum lifetime related deteriorating product.

In the typical economic order quantity model, the demand is taken as a constant and known, but in real life, it is in fact that the demand is variable depending on the selling price and the availability of stock at a particular point of time. Therefore, the demand rate must be considered as a function of either the selling price or the stock level or both. Sana and Chaudhuri (2004) constructed an economic order quantity inventory model by assuming that the demand rate depends on the stock level as well as advertising expenditure. Balkhi and Tadj (2008) studied an economic order quantity model for items with time varying demand and deterioration. Yang et al. (2010) examined an inventory model for deteriorating items in which demand rate depends on item availability with partial backlogging under inflation. To maximize the total profit, using preservation technology for a deteriorating product, Lee and Dye (2012) built an economic order quantity inventory model with partial backlogging and stock-dependent demand. Shaikh et al. (2017) formulated an inventory model according to consideration of price-and stock-dependent demand, fully backlogged shortages and inflation. They also considered deterioration to be non-instantaneous. Mashud et al. (2018) extended the model of Shaikh et al. (2017) by considering the deterioration rate as different constant function in the entire cycle length. Panda et al. (2017) developed an inventory model considering demand as price sensitive and deterioration follows constant rate. Recently, Shaikh et al. (2019) developed an EOQ inventory model for deteriorating items with stock-dependent demand and partial backlogging considering price discount facility. Considering two substitutable products in a two echelon supply chain, Taleizadeh et al. (2019a) addressed a price optimizing inventory model.

At present, it has become a common practice that the vendors offer to the buyers a permissible delay in payment in order to provide them enough flexibility and financial viability to keep the items in store, a procedure for reducing the inventory costs in accordance with their business requirements. On the other hand, buyers are also benefited as they can earn some interest during this credit period on the saved sum due to delayed payment. However, if a retailer does not make payment within the stipulated period, then a higher rate of interest is applied according to terms and conditions. Permissible delay in payment for a single item inventory model was first developed by Goyal(1985). Then, this inventory model was extended for deteriorating items by Aggarwal and Jaggi (1995). In a permissible delay in payment situation, Yadav et al. (2015) investigated an inventory model for a deteriorating product with the effect of inflation in a fuzzy environment. Diabat et al. (2017) developed an inventory model with partial downstream delay in payment, partial upstream advance in payment, and partial backordering for deteriorating items. Panda et al. (2019) incorporated an alternative trade credit policy for a deteriorating product.

A lot of experiments have been undertaken by a number of researchers/academicians to study the concept of permissible delay in payment in inventory management. There are a few numbers of studies concerning the pre-payment system. The stock position of a product may have volatility in the market, due to its high demand and inadequate supply. To adjust to this situation, the buyers want to prepay either the full or partial purchasing cost to vendors to ensure an on-time delivery guarantee of goods. This type of advanced payment model was developed by Zhang (1996) which considers a fixed pre-payment cost. Maiti et al. (2009) observed that the prepayment has a positive effect on the inventory system. They studied an inventory model with advance payment in a stochastic environment, using generalized reduced gradient (GRG) technique and stochastic search genetic algorithm (GA). Thangam (2012) introduced optimal price discount and lot sizing policies for the perishable items under advance payment system. Taleizadeh et al. (2013) analyzed an economic order quantity inventory model with multiple partial prepayments for constantly demanded items. Sarkar et al. (2013) built an inventory model considering both trade credit price discount offers. Saxena et al. (2017) established a green supply chains model for both vender and buyer. Sana et al. (2008) have derived an inventory model with both delay in payments and price discount offers. Pal et al. (2014) have derived a production-inventory model considering permissible delay payments for a three layer supply chain system. Zhang et al. (2014) formulated an EOQ model under advanced payment scheme. Pal et al. (2015) has considered price, quality, and promotional effort sensitive demand and developed for a manufacturer- retailer point of view. Zia and Taleizadeh (2015) investigated a lot sizing model under advanced and delayed payment scheme. Taleizadeh (2014b) revisited and extended the model of Taleizadeh (2014a) by considering a deteriorating product with partially backlogged shortages. Zhang et al. (2016) presented a two-stage supply chain under advanced payment scheme. Lashagari et al. (2016) considered two level trade credit system and established an inventory model. Li et al. (2017) suggested a pricing and lot sizing policy for perishable product and cash flow analysis under advance payment policy. Taleizadeh (2017) analyzed a lot sizing model with advanced payment pricing and disruption in supply chain system. Khan et al. (2018) applied the advanced payment policy for a deteriorating item in a two-warehouse environment and Shaikh et al. (2019) extended the model of Khan et al. (2018) considering all inventory cost in interval form and then solved by particle swarm optimization technique. Shaikh et al. (2019) introduced an EOQ model with price discount inventory model for deteriorating items. Taleizadeh et al. (2019b) developed an inventory model by coordinating different aspects such as segmentation, credit payment and quantity discount offers. In a recent study, Farshbaf-Geranmayeh et al. (2019) discussed the effect of cooperative advertising in the order of a manufacturer–retailer supply chain. Under the effect of product pure bundling on marketing strategies; Eghbali-Zarch et al. (2019) determined the optimal pricing decisions in a centralized three-level supply chain. Shaikh et al. (2019) formulated an inventory model with provision of preservation facility and trade credit. All the aforementioned research works considered the prepayment only for getting an on-time delivery of guarantee of goods but do not contemplate any discount on the purchasing cost. In the reality, when it comes to dealing inthe vendor (company) for an invoice from a buyer, after a certain period it can create a cash flow problem for the vendor from paying merchants, providing workers' wages to other vendors' expenditure, in particular, for a new growing vendor. In this situation, vendor requests to prepay the purchasing cost under some discount facilities.

The present study takes the concept of prepayment discount facility on the purchasing cost based on the amount of prepayment i.e., if a buyer pays his total purchasing cost before receiving the lot, then he gains a certain percentage discount on the total purchasing cost at the time of prepaying. However, if he prepays a portion of the total purchasing cost then the discount is given at the time of receiving the lot by paying the rest part of the total purchasing cost. In this case, the percentage of discount is lower than the percentage in the case of the full amount of prepayment. In this paper, it is implemented this concept for a deteriorating product with price and stock-dependent demand. In the existing literature, most of the authors have considered that the demand is constant. In this work, it is assumed the following features: price and stock-dependent demand, constant deterioration rate and discount facility due to the advance payment. The entire problem is solved theoretically and validated through graphically (by using line diagram and 3D plot) with the help of a numerical example. Table1 presents some related research works and their contribution.

Table 1. Related research works and their contribution

Literature	EOQ/EPQ Model	Payment	Demand rate	Deterioration	Shortage	Discount	Solution methodology
Maiti et al. (2009)	EOQ	Advance	Price-dependent (Non-linear function)	No	Complete backlogging	No	Soft computing
Taleizadeh et al. (2014a)	EOQ	Advance	Constant	Constant	Complete backlogging	No	Mathematically
Taleizadeh et al. (2014b)	EOQ	Advance	Constant	Constant	Partial backlogging	No	Mathematically
Zhang et al. (2016)	EOQ	Advance	Constant	No	No	No	Mathematially
Lashgari et al. (2016)	EOQ	Advance and Delay	Constant	No	No shortages, Complete and Partial backlogging	No	Mathematically
Teng et al. (2016)	EOQ	Advance	Constant	Time-Varying	Partial backlogging	No	Mathematically
Li et al. (2017)	EOQ	Advance and Delay	Price-dependent	Time-Varying	No	No	Mathematically
Taleizadeh (2017)	EOQ	Advance	Constant	No	Partial backlogging	N0	Mathematically
Khan et al. (2018)	EOQ	Advance	Price-dependent	Constant	Partial backlogging	No	Mathematically
Shaikh et al. (2019)	EOQ	Advance	Price-dependent	Constant	Partial backlogging	No	Soft computing
This paper	**EOQ**	**Advance**	**Price and stock-dependent (Linear function)**	**Constant**	**Partial backlogging**	**Yes**	Mathematically

The rest parts of the paper are well thought-out as follows. Section 2 establishes the notation and assumptions. Section 3 defines the inventory models which are formulated mathematically and solved. Section 4 discusses about some special cases. Section 5 provides numerical illustrations. Section 6 presents a sensitivity analysis. Managerial insights are discussed in Section 7. Finally, Section 8 gives some conclusions and future research directions.

2. Problem definition, notation and assumptions

2.1 Problem definition

Consider a situation where early payments discount is offered by the vendor. In this condition; the buyer pays his purchasing cost at a time M before receiving goods. And then he obtains a certain percentage r discount on the total purchasing cost at the time of prepaying. It is clear that to take the advantages of the early payment discount, buyers should have sufficient cash balances. Moreover, some customers run their business with small cash balance and have to borrow money from a bank with a particular interest rate. In the advanced payment system, buyers need to arrange the payable money before receiving the goods. This payable money serves as the capital of the buyers. This money is not always available in the buyers' hand. In this situation, for a smooth running of the business, buyers need to raise money through bank loans or other sources with some interest. This is with an additional cost of their capital. Due to the advanced payment of total purchasing cost of the goods, buyers obtain the discount on purchasing cost (a certain percentage) immediately in cash when he or she prepays. Additionally, sometimes the buyer is unable to prepay the whole purchasing cost of the product. In this case, the buyer wants to prepay a fraction δ of the purchasing cost instead of the full purchasing cost of the goods with the rest to be paid at the time of receiving of goods. In order to keep his customers in hand, the vendor accepts the buyer's proposal and offers some discount which is always less than the amount of discount for prepaying the whole purchasing cost. Here, the discount is given when the buyer receives the goods by paying the rest part. Consequently, according to the amount of prepayment, two cases are possible, namely, case 1: full advance payment of the total purchasing cost and case 2: partial advance payment of the total purchasing cost. For each case, an inventory model is identified and developed for a deteriorating product with partial backlogging shortages.

2.2 Notation

In order to develop the inventory models, the following notation and assumptions are used.

Symbol	Units	Description
K	\$/order	replenishment cost per order
c_i	\$/unit	purchasing cost per unit
c_h	\$/unit/time unit	holding cost per unit per time unit
c_s	\$/unit/time unit	shortages cost per unit per time unit
c_d	\$/unit	deterioration cost per unit
c_l	\$/unit	opportunity cost per unit
p	\$/unit	selling price per unit
θ	constant	deterioration rate
M	Time unit	a time during which the buyer will pay the prepayment
I_e	\$/time unit	cost of loan rate
r	%	discount percentage of the total purchasing cost
S	units	highest stock level
R	units	maximum shortages level
η	constant	backlogging rate
$Q=S+R$	units	order size per cycle
$I(t)$	units	inventory level at any time t where $0 \le t \le T$
$TC(t_1,T)$	\$/time unit	the total cost per unit time
		Decision variables
t_1	Time unit	time at which the stock of the product reaches to zero for all cases with shortages
T	Time unit	the length of the replenishment cycle.

2.3 Assumptions

1. The inventory models are developed for a single deteriorating product for linearly price and stock-dependent demand pattern which is given by $D(p)=\begin{cases} a-bp+cI(t) & \text{when } I(t)\ge 0 \\ a-bp & \text{when } I(t)<0 \end{cases}$

   It is important to remark that demand function depends on price and stock, while during the shortage time, demand depends only on the price of the product. It is assumed that $a-bp\ge 0$.

2. The deterioration rate $\theta\,(0<\theta<<1)$ is constant.
3. There is no replacement or repair for deteriorated products during the period under consideration.
4. Replenishment rate is infinite and lead time is M .
5. The total planning horizon of the inventory system is infinite.
6. When the buyer pays his purchasing cost at time M before receiving goods, he obtains a certain percentage r discount of the total purchasing cost at the time of prepaying.However, when he prepays a fraction $\delta(0<\delta<1)$ of the total purchasing cost discount, that must be less than the case of full prepayment, will be given at the receiving time of a lot by payingthe rest part.
7. Shortages are allowed with a constant backlogging rate η.

3. Model formulation and solution procedure

Based on the buyer's advance payment amount, full or fraction, there are two cases in this inventory problem. First, it is delineated the inventory model under full prepayment of the total purchasing cost and then under partial prepayment of the total purchasing cost.

3.1 Case 1: Full advance payment of the total purchasing cost

In this case, the buyer purchases the products by prepaying the full purchasing cost before receiving a lot. Initially, a buyer purchases goods, $Q=(S+R)$ units, from a vendor with an early payment discount, by paying the full purchasing cost at a time M before receiving goods. Due to meet up the customers' demand D as well as the effect of the deterioration, the stock is depleted and at time $t=t_1$ stock becomes zero. Then shortages appear which are partially backlogged with a rate η (see Fig. 1). Thus, this situation is modeled by the following differential equations:

$$\frac{dI_1(t)}{dt}+\theta I(t)=-(a-bp+cI(t)), \qquad 0\le t\le t_1 \tag{1}$$

with the boundary conditions $I_1(0)=S$ and $I_1(t_1)=0$.

$$\frac{dI_2(t)}{dt}=-\eta(a-bp), \qquad t_1\le t\le T \tag{2}$$

with the boundary conditions $I_2(t_1)=0$ and $I_2(T)=-R$.

Using boundary condition $I_1(t_1)=0$ from (1), one has:

$$I_1(t)=\frac{a-bp}{\theta+c}\left\{e^{(\theta+c)(t_1-t)}-1\right\}. \tag{3}$$

Now, with the help of the boundary condition $I_1(0)=S$, the maximum stock level is:

$$S=\frac{a-bp}{\theta+c}\left\{e^{(\theta+c)t_1}-1\right\}. \tag{4}$$

Applying Taylor series expansion for the exponential term and neglecting higher order terms of θ and c, because in reality they are very insignificant, hence:

$$S \approx (a-bp)\left[t_1 + \frac{(\theta+c)t_1^{\,2}}{2}\right]. \tag{5}$$

From (2) with $I_2(t_1)=0$, anyone has:

$$I_2(t) = \eta(a-bp)(t_1 - t). \tag{6}$$

With the boundary condition $I_2(T) = -R$, the maximum shortages level is:

$$R = \eta(a-bp)(T-t_1). \tag{7}$$

Thus, the total ordering quantity is:

$$Q = (a-bp)\left[t_1 + \frac{(\theta+c)t_1^{\,2}}{2} + \eta(T-t_1)\right]. \tag{8}$$

Therefore, the total purchasing cost for the buyer is calculated with $c_i(a-bp)\left[t_1 + \frac{(\theta+c)t_1^{\,2}}{2} + \eta(T-t_1)\right]$. Since the buyer prepays the full purchasing cost at the time M before receiving a lot, he obtains a certain percentage r discount of the total purchasing cost at the time of prepaying. As a result, the reduced purchasing cost and the corresponding cost of loan are determined as follows $(1-r)c_i(a-bp)\left[t_1 + \frac{(\theta+c)t_1^{\,2}}{2} + \eta(T-t_1)\right]$ and $I_e M(1-r)c_i(a-bp)\left[t_1 + \frac{(\theta+c)t_1^{\,2}}{2} + \eta(T-t_1)\right]$ respectively (see Figure 1). The components of the total cost are:

(a) Ordering cost: K

(b) Purchasing cost: $(1-r)c_i(a-bp)\left[t_1 + \frac{(\theta+c)t_1^{\,2}}{2} + \eta(T-t_1)\right]$

(c) Holding cost: $c_h\int_0^{t_1} I_1(t)\,dt = c_h\frac{a-bp}{(\theta+c)^2}\left\{e^{(\theta+c)t_1} - (\theta+c)t_1 - 1\right\} \approx \frac{1}{2}c_h(a-bp)t_1^{\,2}$

(d) Deterioration cost: $\theta c_d\int_0^{t_1} I_1(t)\,dt = \theta c_d\frac{a-bp}{(\theta+c)^2}\left\{e^{(\theta+c)t_1} - (\theta+c)t_1 - 1\right\} \approx \frac{1}{2}\theta c_d(a-bp)t_1^{\,2}$

(e) Cost of loan: $I_e M(1-r)c_i(a-bp)\left[t_1 + \frac{(\theta+c)t_1^{\,2}}{2} + \eta(T-t_1)\right]$

(f) Shortage cost: $-c_s\int_{t_1}^{T} I_2(t)\,dt = \frac{1}{2}c_s(a-bp)(T-t_1)^2$

(g) Opportunity cost: $c_l(1-\eta)\int_{t_1}^{T} D\,dt = c_l(1-\eta)(a-bp)\left(T-t_1\right)$.

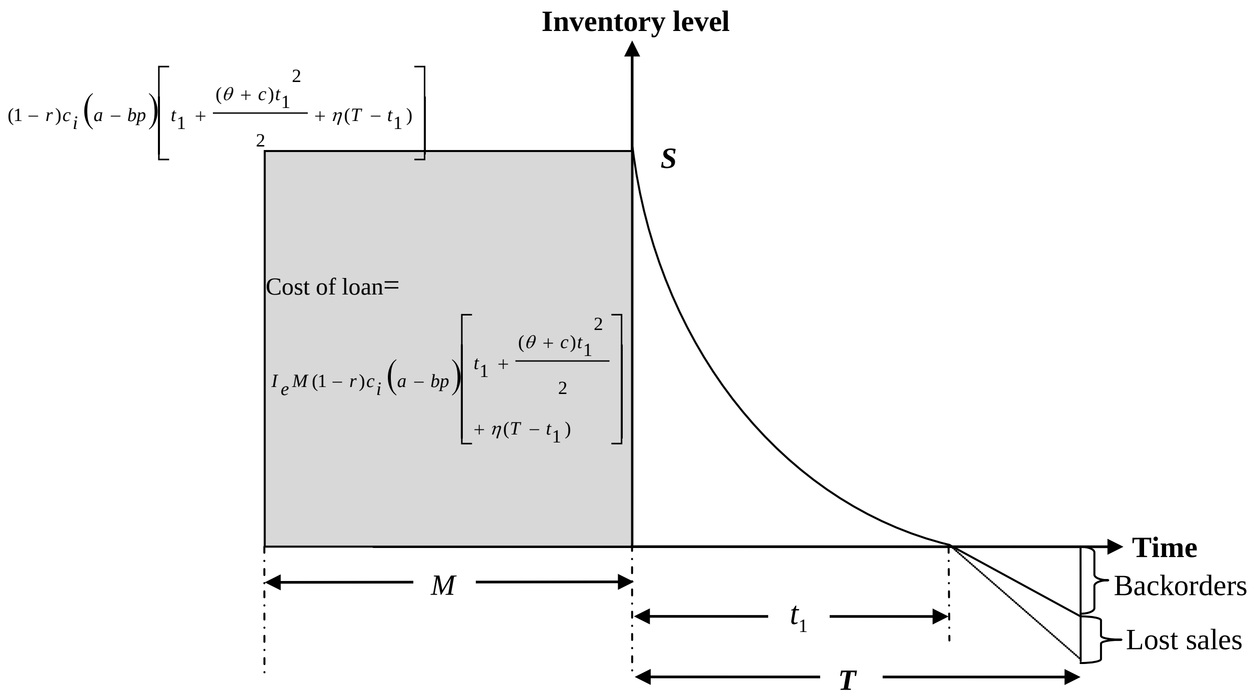

Figure 1.Graphical presentation of the inventory system with full prepayment when shortages are partially backlogged.

The total cost in this case is:

$$TC = K+(1+I_eM)(1-r)c_i\left(a-bp\right)\left[t_1+\frac{(\theta+c)t_1^2}{2}+\eta(T-t_1)\right]+\frac{1}{2}(c_h+\theta c_d)(a-bp)t_1^2 + \frac{1}{2}c_s\eta(a-bp)(T-t_1)^2+c_l(1-\eta)(a-bp)(T-t_1). \tag{9}$$

Hence, the total cost per unit time is:

$$TC(t_1,T)=\frac{1}{T}\left[\begin{array}{l} K+(1+I_eM)(1-r)c_i\left(a-bp\right)\left[t_1+\frac{(\theta+c)t_1^2}{2}+\eta(T-t_1)\right] \\ +\frac{1}{2}(c_h+\theta c_d)(a-bp)t_1^2+\frac{1}{2}c_s\eta(a-bp)(T-t_1)^2+c_l(1-\eta)(a-bp)(T-t_1) \end{array}\right]. \tag{10}$$

From the assumptions, we have $t_1 < T$. Hence, for simplicity of the solution procedure, we can take

$$t_1=\alpha T,\ \ 0<\alpha<1. \tag{11}$$

Using Eq. (11) in Eq. (10), we obtain:

$$TC(\alpha,T)=\begin{bmatrix}\frac{K}{T}+(1+I_eM)(1-r)c_i\left(a-bp\right)\left[\alpha+\frac{(\theta+c)\alpha^2T}{2}+\eta(1-\alpha)\right]\\+\frac{1}{2}(c_h+\theta c_d)(a-bp)\alpha^2T+\frac{1}{2}c_s\eta(a-bp)(1-\alpha)^2T+c_l(1-\eta)(a-bp)(1-\alpha)\end{bmatrix}. \tag{12}$$

The total cost per unit time $TC(\alpha,T)$ is written as:

$$TC(\alpha,T)=\frac{\varphi_1}{T}+\left(\varphi_2\alpha^2-2\varphi_3\alpha+\varphi_3\right)T+\varphi_4\alpha+\varphi_5 . \text{ [See Appendix A for all values]} \tag{13}$$

Equation (13) is expressed as:

$$TC(\alpha,T)=\frac{\varphi_1}{T}+F_1(\alpha)T+F_2(\alpha) \tag{14}$$

where $F_1(\alpha)=\varphi_2\alpha^2-2\varphi_3\alpha+\varphi_3, F_2(\alpha)=\varphi_4\alpha+\varphi_5$ (15)

and so

$$TC(\alpha,T)=\frac{\left(T\sqrt{F_1(\alpha)}-\sqrt{\varphi_1}\right)^2}{T}+2\sqrt{F_1(\alpha)}\sqrt{\varphi_1}+F_2(\alpha) . \tag{16}$$

The total cost per unit time $TC(\alpha,T)$ is minimum with respect to T if

$$T=\sqrt{\frac{\varphi_1}{F_1(\alpha)}} . \tag{17}$$

It is clear that the value of T cannot be obtained if $F_1(\alpha)$ is either zero or negative.

Theorem 1. The function $F_1(\alpha)$ is strictly positive.

Proof: See Appendix B.

Consequently, from Eq. (17) we get a positive value of T for which cost will be minimum. With the help of the value of T, the Eq. (16) becomes

$$TC(\alpha)=2\sqrt{F_1(\alpha)}\sqrt{\varphi_1}+F_2(\alpha) . \tag{18}$$

Theorem 2. The reduced total cost function $TC(\alpha)$ is minimized at $\alpha=\frac{2T\varphi_3-\varphi_4}{2T\varphi_2}$.

Proof: See the Appendix C.

Finally, from the Eq. (17), the optimal value of the cycle length is

$$T^*=\sqrt{\frac{4\varphi_1\varphi_2-\varphi_4{}^2}{4(\varphi_2\varphi_3-\varphi_3{}^2)}} . \tag{19}$$

With the help of Theorem (2) and Eq. (19), from Eq. (11) we get

$$t_1^* = \frac{\varphi_3}{\varphi_2}\sqrt{\frac{4\varphi_1\varphi_2 - \varphi_4^{\,2}}{4(\varphi_2\varphi_3 - \varphi_3^{\,2})}} - \frac{\varphi_4}{2\varphi_2}. \tag{20}$$

Theorem 3. If $\eta > 1 - \sqrt{\dfrac{2K\{(1+I_eM)(1-r)c_i(\theta+c)+c_h+\theta c_d\}}{(a-bp)\{(1+I_eM)(1-r)c_i - c_l\}^2}}$, then the solution (t_1, T), from equations (19) and (20), optimizes the cost function (10).

Proof: See the Appendix D.

If $\eta < 1 - \sqrt{\dfrac{2K\{(1+I_eM)(1-r)c_i(\theta+c)+c_h+\theta c_d\}}{(a-bp)\{(1+I_eM)(1-r)c_i - c_l\}^2}}$, then the model with partial backlogging under full prepayment of the total purchasing cost does not provide the optimal cost for the buyer. In this situation, the optimal solution is found either in the model without any shortage or not to store the product. Next the model is delineated under the partial advance payment of the total purchase cost.

3.2 Case 2: Partial advance payment of the total purchasing cost

Sometimes the buyer cannot prepay the whole purchasing cost. In this case, vendor gives an opportunity to his buyer to prepay a fraction $\delta\,(0 < \delta < 1)$ of the purchasing cost with a less discount (than the full advance payment) and allowstopaythe rest part at the receiving time of a lot. But in this case, the discount is given when the buyer receives the lot by paying therest part.Here, the buyer purchases goods, $Q=(S+R)$ units, from a vendor by prepaying a fraction δ of the purchasing cost at a time M before getting goods and receives the lot by paying the remaining portion of the total purchasing cost after deducting the discount amount (see Figure 2). Since the behavior of the inventory levels over time for this case and the C*ase 3.1* are same, all the inventory-related costs in this case are identical with *Case 3.1* except the cost of loan. For this case, the corresponding cost of loan is obtained as follows: $I_eM\delta c_i(a-bp)\left[t_1 + \dfrac{(\theta+c)t_1^{\,2}}{2} + \eta(T-t_1)\right]$. Thus the total cost per unit time is:

$$TC(t_1,T) = \frac{1}{T}\begin{bmatrix} K + (1+I_eM\delta - r)c_i(a-bp)\left[t_1 + \dfrac{(\theta+c)t_1^{\,2}}{2} + \eta(T-t_1)\right] \\ + \dfrac{1}{2}(c_h+\theta c_d)(a-bp)t_1^{\,2} + \dfrac{1}{2}c_s\eta(a-bp)(T-t_1)^2 + c_l(1-\eta)(a-bp)(T-t_1) \end{bmatrix}. \tag{21}$$

Proceeding on the similar mode as we have utilized in the C*ase 3.1*, the optimal values of t_1 and cycle's length for this case are:

$$t_1^* = \frac{\varphi_3}{\varphi_2}\sqrt{\frac{4\varphi_1\varphi_2 - \varphi_4^{\,2}}{4(\varphi_2\varphi_3 - \varphi_3^{\,2})}} - \frac{\varphi_4}{2\varphi_2}, \tag{22}$$

$$T^* = \sqrt{\frac{4\varphi_1\varphi_2 - \varphi_4{}^2}{4(\varphi_2\varphi_3 - \varphi_3{}^2)}}\,, \tag{23}$$

where

$$\varphi_1 = K\,,\quad \varphi_2 = \frac{1}{2}\left[(1+I_e M\delta - r)c_i(\theta + c) + (c_h + \theta c_d + c_s\eta)\right]\!\left(a - bp\right),\quad \varphi_3 = \frac{1}{2}c_s\eta(a-bp)\,,$$

$$\varphi_4 = \left\{(1+I_e M\delta - r)c_i - c_l\right\}(1-\eta)\left(a-bp\right), \varphi_5 = \left\{(1+I_e M\delta - r)c_i\eta + c_l(1-\eta)\right\}\left(a-bp\right).$$

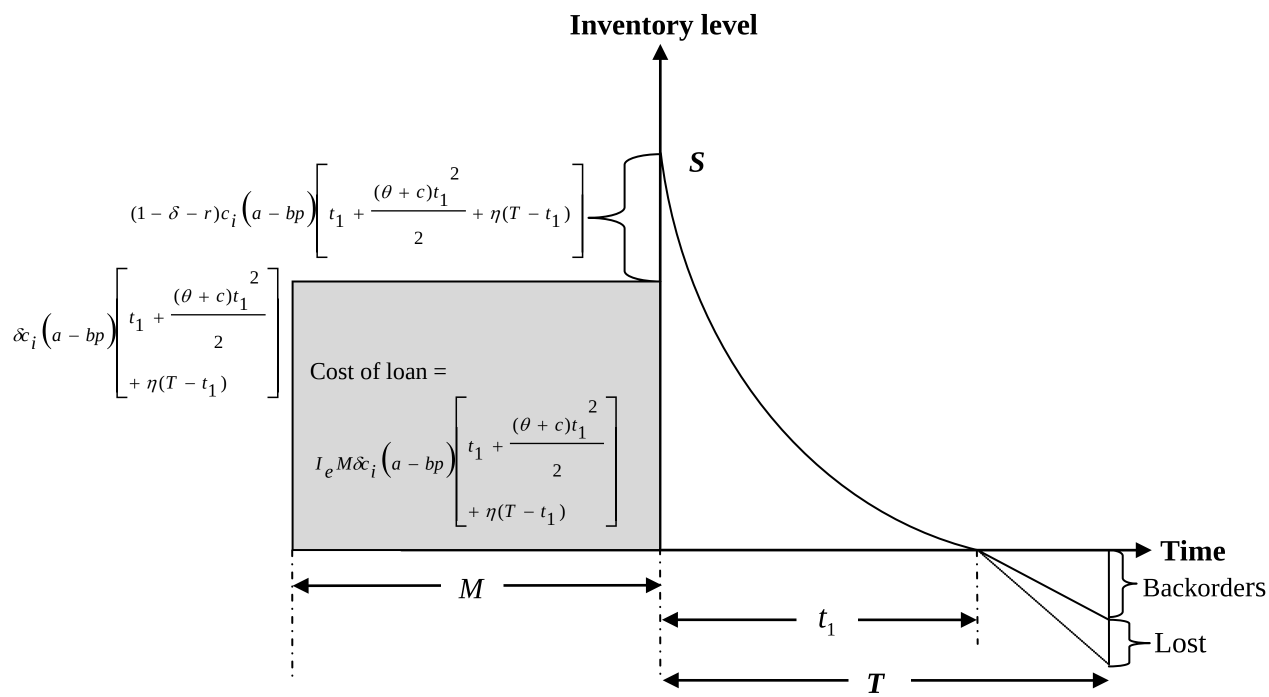

Figure 2.Graphical presentation of the inventory system with partial prepayment when shortages are partially backlogged.

Theorem 4. If $\eta > 1 - \sqrt{\dfrac{2K\{(1+I_e M\delta - r)c_i(\theta+c) + c_h + \theta c_d\}}{(a-bp)\{(1+I_e M\delta - r)c_i - c_l\}^2}}$, then the solution (t_1, T), from equations (22) and (23), optimizes the cost function (21).

Proof: Similar to the proof of Theorem 5.

If $\eta < 1 - \sqrt{\dfrac{2K\{(1+I_e M\delta - r)c_i(\theta+c) + c_h + \theta c_d\}}{(a-bp)\{(1+I_e M\delta - r)c_i - c_l\}^2}}$, then the model with partial backlogging under partial prepayment of the total purchasing cost does not provide the optimal cost. Then, the optimal solution is found either in the model without any shortage or not to store the product.

4. Special cases

It is noteworthy that, if $\eta = 1$, the proposed model with partial backlogging becomes the model with fully backlogging and also if $\eta = 0$ and $T = t_1$, then the model becomes the model without any shortages.

4.1. Full advance payment of the total purchasing cost for fully backlogged shortages

For $\eta = 1$, the total cost is given as:

$$TC(t_1,T) = \frac{K}{T} + (1+I_eM)(1-r)c_i\left(a-bp\right)\frac{1}{T}\left[T + \frac{(\theta+c)t_1^2}{2}\right] + \frac{1}{2}(c_h+\theta c_d)(a-bp)\frac{t_1^2}{T}$$
$$+\frac{1}{2}c_s(a-bp)\frac{(T-t_1)^2}{T}.$$

From the above objective function, we can found the optimal values of t_1 and T which are given below:

$$t_1^* = \frac{c_s}{(1+I_eM)(1-r)c_i(\theta+c)+c_h+\theta c_d+c_s}\sqrt{\frac{2K\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d+c_s)\right]}{\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right]c_s\left(a-bp\right)}},$$

and $T^* = \sqrt{\frac{2K\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d+c_s)\right]}{\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right]c_s\left(a-bp\right)}}$.

The corresponding optimal total cost per unit is:

$$TC^* = \sqrt{\frac{2K\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right]c_s\left(a-bp\right)}{\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d+c_s)\right]}} + (1+I_eM)(1-r)c_i\left(a-bp\right).$$

4.2. Partial advance payment of the total purchasing cost for fully backlogged shortage

For $\eta = 1$, the total cost is given as:

$$TC(t_1,T) = \frac{K}{T} + (1+\delta I_eM-r)c_i\left(a-bp\right)\frac{1}{T}\left[T + \frac{(\theta+c)t_1^2}{2}\right] + \frac{1}{2}(c_h+\theta c_d)(a-bp)\frac{t_1^2}{T}$$
$$+\frac{1}{2}c_s(a-bp)\frac{(T-t_1)^2}{T}.$$

From the above objective function, we can found the optimal values of t_1, T and the total cost per unit time which are given below:

$$t_1^* = \frac{c_s}{(1+\delta I_eM-r)c_i(\theta+c)+c_h+\theta c_d+c_s}\sqrt{\frac{2K\left[(1+\delta I_eM-r)c_i(\theta+c)+(c_h+\theta c_d+c_s)\right]}{\left[(1+\delta I_eM-r)c_i(\theta+c)+(c_h+\theta c_d)\right]c_s\left(a-bp\right)}},$$

$$T^* = \sqrt{\frac{2K\left[(1+\delta I_eM-r)c_i(\theta+c)+(c_h+\theta c_d+c_s)\right]}{\left[(1+\delta I_eM-r)c_i(\theta+c)+(c_h+\theta c_d)\right]c_s\left(a-bp\right)}},$$

and $TC^* = \sqrt{\frac{2K\left[(1+\delta I_eM-r)c_i(\theta+c)+(c_h+\theta c_d)\right]c_s\left(a-bp\right)}{\left[(1+\delta I_eM-r)c_i(\theta+c)+(c_h+\theta c_d+c_s)\right]}} + (1+\delta I_eM-r)c_i\left(a-bp\right).$

4.3. Full advance payment of the total purchasing cost for without shortages model

For $\eta = 0$ and $T = t_1$, the total cost is given as:

$$TC(T) = \frac{K}{T} + (1 + I_e M)(1 - r)c_i\left(a - bp\right)\left[1 + \frac{(\theta + c)T}{2}\right] + \frac{1}{2}(c_h + \theta c_d)(a - bp)T.$$

Optimizing the above function, the total cost per unit time TC has a global minimum value at $T^* = \sqrt{\frac{2K}{\left\{(1 + I_e M)(1 - r)c_i(\theta + c) + (c_h + \theta c_d)\right\}\left(a - bp\right)}}$.

4.4. Partial advance payment of the total purchasing cost for without shortages model

Therefore, for $\eta = 0$ *and* $t_1 = T$, the total cost per unit time for the model without shortages is:

$$TC(T) = \frac{K}{T} + (1 + I_e M\delta - r)c_i(a - bp)\left[1 + \frac{(\theta + c)T}{2}\right] + \frac{1}{2}(c_h + \theta c_d)(a - bp)T.$$

Optimizing the above function, we can found the optimal cycle length which is:

$$T^* = \sqrt{\frac{2K}{\left\{(1 + I_e M\delta - r)c_i(\theta + c) + (c_h + \theta c_d)\right\}\left(a - bp\right)}}.$$

To examine now the developed inventory models, two numerical examples are carried out in the next section.

5. Numerical illustrations

In this section two numerical examples are solved in order to demonstrate and validate the inventory models; all based on ones from Taleizadeh(2014b), with the additional parameters defined in this paper. The fixed parameters in all examples are K=1,000,000 IR/order, a=250,600 liter/month, b=1.5, p=400 IR/liter, $c_i = 300$ IR/liter/month, $c_h = 30$ IR/liter/month, $c_s = 50$ IR/liter/month, $c_l = 60$ IR/liter/month, $c_d = 40$ IR/liter/month, M=0.25 month, $I_e = 0.3/$ IR/month, $c = 0.2$, $\theta = 0.005$ and $\eta = 0.95$.

At first, we have analyzed the solutions of the modelunder *Case 1* i.e., full advance payment of the total purchasing cost by considering the corresponding discount rate $r = 35\%$.

Example 1: For Case 1

Here, $1 - \sqrt{\frac{2K\{(1 + I_e M)(1 - r)c_i(\theta + c) + c_h + \theta c_d\}}{(a - bp)\{(1 + I_e M)(1 - r)c_i - c_l\}^2}} = 0.838297$

Since, $0.95 = \eta > 1 - \sqrt{\frac{2K\left[(1 + I_e M)(1 - r)c_i(\theta + c) + (c_h + \theta c_d)\right]}{\left(a - bp\right)\left\{(1 + I_e M)(1 - r)c_i - c_l\right\}^2}} = 0.838297$, by Theorem 3 the optimal values of T and t_1 are determined by the following way. The values of $\varphi_i\,(i = 1,2,\ldots,5)$ are as follows:

$\varphi_1 = 1{,}000{,}000$; $\varphi_2 = 15{,}084{,}140$; $\varphi_3 = 5{,}937{,}500$; $\varphi_4 = 1{,}870{,}313$ and $\varphi_5 = 50{,}535{,}940$.

So, from Eq. (19) and Eq. (20), we have $T^* = 0.51152$ and $t_1^* = 0.13935$. Finally, the total cost per unit time for this case is $TC^* = 54,955,410$. The total order quantity and backorder quantity are: $S^* = 123,724.4$ and $R^* = 88,389.41$. Hence, the optimal solution for this subcase is $\{t_1^* = 0.13935$ month, $T^* = 0.51152$ month, $TC^* = 54,955,410$ IR/month$\}$ (see Figure 3). Also, the optimal solution is easily observed from the line diagrams of total cost per unit time versus t_1 and total cost per unit time versus cycle length T in Figure 4 and Figure 5 respectively.

Now we determine the solution of the model under *Case 2* i.e., partial advance payment of the total purchasing cost by considering $\delta = 0.6$ and the corresponding discount rate $r = 20\%$.

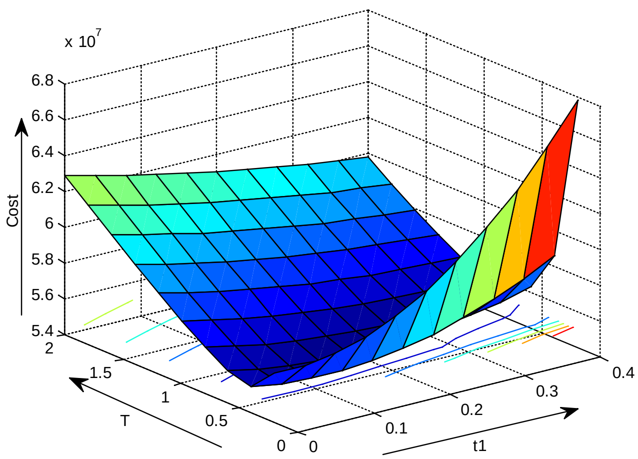

Figure 3. Total cost per unit time versus t_1 and T.

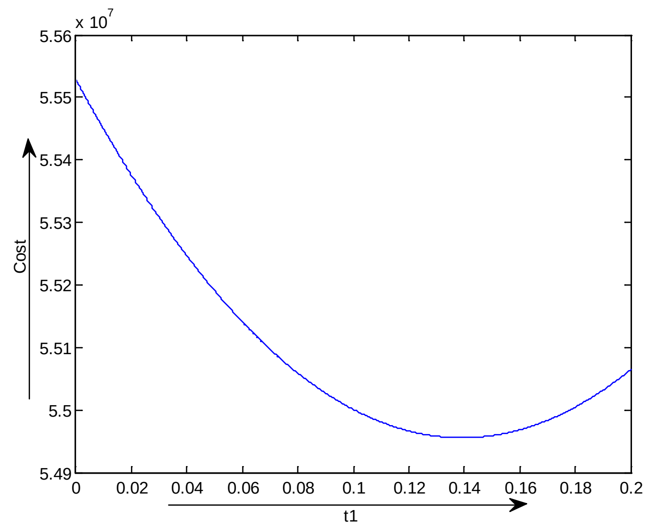

Figure 4. Line diagram of total cost per unit time versus t_1.

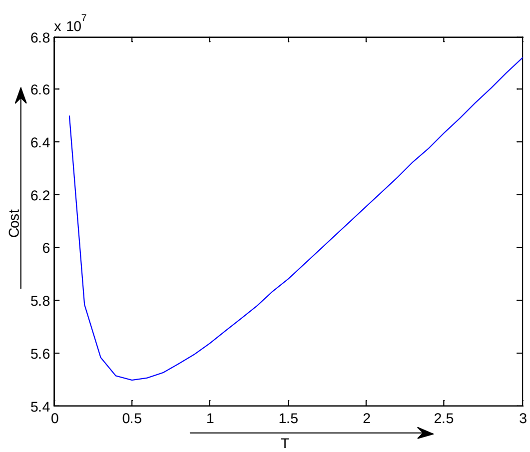

Figure 5. Line diagram of total cost per unit time versus T.

Example 2: For case 2

Here, $1-\sqrt{\frac{2K\{(1+I_eM\delta-r)c_i(\theta+c)+c_h+\theta c_d\}}{(a-bp)\{(1+I_eM\delta-r)c_i-c_l\}^2}}=0.87401$. Since, $0.95=\eta>0.87401$, so by Theorem 4 the optimal values of T and t_1 are computed in the following way. The values of $\varphi_i\,(i=1,2,...,5)$ are given below.

$\varphi_1=1{,}000{,}000$; $\varphi_2=16{,}208{,}440$; $\varphi_3=5{,}937{,}500$; $\varphi_4=2{,}418{,}750$ and $\varphi_5=60{,}956{,}250$.

From Eq. (22) and Eq. (23), we have $t_1^*=0.10552$ and $T^*=0.49173$. Finally, the total cost per unit time for this case is $TC^*=65{,}542{,}540$. The total order quantity and backorder quantity are: $S^*=118{,}390.5$ and $R^*=91725.73$ .Hence, the optimal solution for this subcase is {$t_1^*=0.10552$ month, $T^*=0.49173$ month, $TC^*=65{,}542{,}540$ IR/month} (see Figure 6) . The optimal solution is shown by drawing line diagrams of total cost per unit time versus t_1 and total cost per unit time versus cycle length T in Figure 7 and Figure 8 respectively.

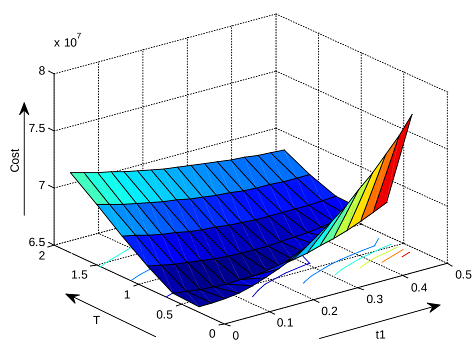

Figure 6. Total cost per unit time versus t_1 and T for 60% early payment.

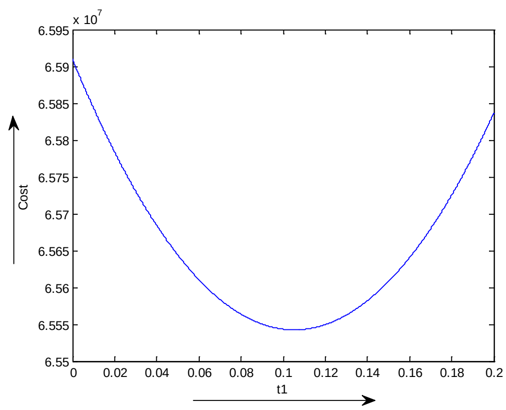

Figure 7. Line diagram of total cost per unit time versus t_1 for 60% early payment.

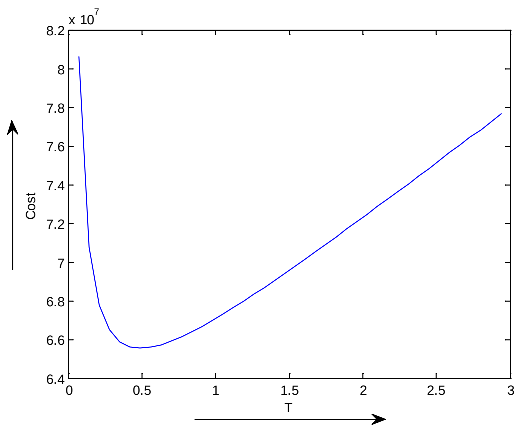

Figure 8.Line diagram of total cost per unit time versus T for 60% early payment.

6. Sensitivity analysis

To observe the impact of the changes of the value of key parameters on the optimal solution, a sensitivity analysisis performed based on the above example 1 for the model with partially backlogged shortages by constructing Table 2 which reveals how much variation on optimal values when each parameter is updated at a time in certain percentage. This analysis has been carried out by changing the value of each parameter -40% to 40% but one parameter at a time and remaining all parameters at their initial values.

Table 2. Sensitivity analysis of the parameters on the optimal solution of example 1.

Parameter	% of changes	Optimal values			% of changes in		
		t_1^*	T^*	TC^*	t_1^*	T^*	TC^*
K	+40	0.17832	0.61053	55668390	27.97	19.36	1.30
	+20	0.15970	0.56320	55327600	14.60	10.10	0.68
	-20	0.11670	0.45398	54541110	-16.25	-11.25	-0.75
	-40	0.09073	0.38800	54066040	-34.89	-24.15	-1.62
a	+40	0.10600	0.42680	76135690	-23.93	-16.56	38.54
	+20	0.12063	0.46397	65561830	-13.43	-9.30	19.30
	-20	0.16457	0.57558	44306780	18.10	12.52	-19.38
	-40	0.20134	0.66900	33599790	44.48	30.79	-38.86
b	+40	0.13945	0.51178	54904530	0.07	0.05	-0.09
	+20	0.13940	0.51165	54929970	0.04	0.03	-0.05
	-20	0.13930	0.51139	54980850	-0.04	-0.03	0.05
	-40	0.13925	0.51125	55006290	-0.07	-0.05	0.09
p	+40	0.13945	0.51178	54904530	0.07	0.05	-0.09
	+20	0.13940	0.51165	54929970	0.04	0.03	-0.05
	-20	0.13930	0.51139	54980850	-0.04	-0.03	0.05
	-40	0.13925	0.51125	55006290	-0.07	-0.05	0.09
θ	+40	0.13853	0.51090	54957760	-0.59	-0.12	0.0043
	+20	0.13894	0.51121	54956590	-0.29	-0.06	0.0021
	-20	0.13976	0.51183	54954220	0.30	0.06	-0.0022
	-40	0.14018	0.51214	54953030	0.59	0.12	-0.0043
c	+40	0.11637	0.49423	55023010	-16.49	-3.38	0.12
	+20	0.12682	0.50206	54991960	-8.99	-1.85	0.07
	-20	0.15469	0.52318	54911740	11.01	2.28	-0.08
	-40	0.17392	0.53793	54858590	24.81	5.16	-0.18
c_i	+40	0.07883	0.47456	75149560	-43.43	-7.23	36.75
	+20	0.10691	0.49259	65073010	-23.28	-3.70	18.41
	-20	0.17735	0.53183	44788250	27.27	3.97	-18.50
	-40	0.22265	0.55430	34559950	59.78	8.36	-37.11
c_h	+40	0.12209	0.49851	55005940	-12.39	-2.54	0.09
	+20	0.13014	0.50456	54982180	-6.61	-1.36	0.05
	-20	0.14999	0.51959	54925000	7.63	1.58	-0.06
	-40	0.16242	0.52909	54890160	16.55	3.44	-0.12
c_d	+40	0.13922	0.51142	54955790	-0.09	-0.02	0.0007
	+20	0.13928	0.51147	54955600	-0.05	-0.01	0.0003
	-20	0.13942	0.51157	54955220	0.05	0.01	-0.0003
	-40	0.13948	0.51162	54955030	0.09	0.02	-0.0007
c_s	+40	0.16880	0.46704	55494190	21.14	-8.69	0.98
	+20	0.15537	0.48607	55248420	11.49	-4.98	0.53
	-20	0.11984	0.54747	54598480	-14.00	7.03	-0.65
	-40	0.09535	0.60266	54150500	-31.58	17.82	-1.46
c_l	+40	0.15111	0.51614	55170610	8.44	0.90	0.39
	+20	0.14527	0.51393	55063770	4.25	0.47	0.20
	-20	0.13334	0.50888	54845490	-4.31	-0.51	-0.20
	-40	0.12725	0.50603	54734010	-8.69	-1.07	-0.40

M	+40	0.13452	0.50880	56369920	-3.46	-0.53	2.5739
	+20	0.13692	0.51016	55662780	-1.74	-0.27	1.2872
	-20	0.14180	0.51288	54247790	1.76	0.27	-1.2876
	-40	0.14428	0.51425	53539940	3.54	0.54	-2.5757
I_e	+40	0.13452	0.50880	56369920	-3.46	-0.53	2.5739
	+20	0.13692	0.51016	55662780	-1.74	-0.27	1.2872
	-20	0.14180	0.51288	54247790	1.76	0.27	-1.2876
	-40	0.14428	0.51425	53539940	3.54	0.54	-2.5757
r	+40	0.18055	0.53347	44003800	29.57	4.29	-19.9282
	+20	0.15902	0.52224	49487600	14.12	2.10	-9.9495
	-20	0.12128	0.50119	60408900	-12.97	-2.02	9.9235
	-40	0.10461	0.49117	65849490	-24.93	-3.98	19.8235

From the above Table 2, the following features are concluded:

(i) The total cost (TC^*) is highly sensitive with respect to the demand parameter a, discount rate r and purchasing cost per unit c_i i.e., if the mentioned parameters' values increase then total cost decreases rapidly and vice-versa. The total cost is moderately sensitive with respect to $K, p, c, b, c_h, c_s, c_l, I_e$ and M. It indicates the total cost gradually increases or decreases if the mentioned parameters values change. On the other hand, the total cost remains static with respect to the parameters θ and c_d .This means that, if the said parameters values are changed then it has hardly any effect on the total cost of the inventory system.

(ii) Cycle length (T^*) of the inventory model is highly sensitive with respect to the parameters K, a, discount rate r and c_s . On the other side, cycle length is moderately sensitive with respect to I_e, c, c_i, c_s M and c_l . It specifies that a change of the cycle length (increases or decreases) is occurred slowly if the said parameters values are modified.Butthe cycle length is less sensitive with respect to the parameters b, p, θ and c_d .

(iii) The time at which the inventory level reaches to zero (t_1^*) is highly sensitive with regard to the parameters $K, a, c, c_h, c_i, c_s, r$ and c_l whereas t_1^* is moderately sensitive respecting to the parameters θ, M , I_e and less sensitive with respect to the rest of the parameters.

7. Managerial Insights

In accordance with the performed sensitivity analysis, the following findings are recommended to the decision maker in order to decrease the total cost per unit time.

- As the discount rate on the total purchasing under full advance payment is greater than the discount under partial advance payment of the total purchasing cost, the total cost under the full advance payment scheme is lower than that of under the partial advance payment scheme. Therefore, it is exhorted to the decision make to prefer always the full advance payment policy.

- When the discount rate due to the advance payment increases, the total cost reduces. Consequently, the decision maker is advised to select the vendor or supplier who offers a higher discount rate or persuade the vendor or supplier to wax the discount rate.
- If the cost of loan rate waxes, the total cost per unit time also waxes. Thus, the manager ought to choose the vendor or supplier who allows a small cost of loan rate.
- When the time period during which the buyerpays the prepayment subsides, the total loan cost also subsides and as a result the cost per unit time wanes. So the decision maker should select the vendor or supplier who offers a small time period for the prepayment.
- When the unit purchase cost decreases, the total cost per unit time also decreases significantly. So, reducing the unit purchase cost in another recommendation for reducing the total cost to the decision maker by negotiating with thevendor or supplier.Also, the decision maker is advised to take the proper marketing strategies in order to the customers' demand.

8. Conclusion

This paper presents an inventory model for deteriorating products with advance payment and discount facility due to advance payment. Here, it is considered two different situationsbased on the prepayment amount, full or partial, of the total purchasing cost. Under both situations, the inventory model with shortages is described mathematically. Also, it is shown the convexity of total cost function theoreticallyand graphically as well. The total cost under the full advance payment scheme is lower than that of under the partial advance payment scheme due to a higher discount rateon the total purchasing under full advance payment. So, it is recommended to the buyer to select the full advance payment policy. A sensitivity analyses is conducted to study the relation between different parameters and the optimal solution. It is clear from these analyses that the demand parameter a and the unit purchasing cost c_i play a principal role on the total cost which exposes that the decision maker (buyer) must allay the total cost by implementing proper marketing policies for boosting customers' demand and also by negotiating with suppliers to reduce the unit purchasing cost as much as possible.

Finally, the proposed inventory model can be extended by considering several realistic features. A constant backlogging rate is considered in the current inventory model, so one interesting extension is possible by taking into account partial backlogging at a rate with the length of the waiting time to the arrivals of next lot. One may also extend this inventory model by considering nonlinear demand pattern with nonlinear holding cost. Also, trade credit (single level, two level or partial) can be explored.

Appendix A.

$$\varphi_1 = K\,, \varphi_2 = \frac{1}{2}\left[(1+I_e M)(1-r)c_i(\theta + c) + (c_h + \theta c_d + c_s\eta)\right](a - bp),\ \varphi_3 = \frac{1}{2}c_s\eta(a-bp)\,,$$

$$\varphi_4 = \{(1+I_e M)(1-r)c_i - c_l\}(1-\eta)(a-bp), \varphi_5 = \{(1+I_e M)(1-r)c_i\eta + c_l(1-\eta)\}(a-bp).$$

Appendix B. Proof of Theorem 1

$$F_1(\alpha)=\varphi_2\alpha^2-2\varphi_3\alpha+\varphi_3=\varphi_2\left(\alpha^2-2\alpha\frac{\varphi_3}{\varphi_2}\right)+\varphi_3=\varphi_2\left(\alpha-\frac{\varphi_3}{\varphi_2}\right)^2-\frac{\varphi_3{}^2}{\varphi_2}+\varphi_3=\varphi_2\left(\alpha-\frac{\varphi_3}{\varphi_2}\right)^2+\frac{\varphi_2\varphi_3-\varphi_3{}^2}{\varphi_2}.$$

If $\varphi_2\varphi_3-\varphi_3{}^2>0$, then $F_1(\alpha)$ is always positive. Thus,

$$\varphi_2\varphi_3-\varphi_3{}^2=\frac{1}{2}\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d+c_s\eta)\right](a-bp).\frac{1}{2}c_s\eta(a-bp)-\frac{1}{4}c_s{}^2\eta^2(a-bp)^2$$

$$\phi_2\phi_3-\phi_3^2=\frac{1}{4}\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right]c_s\eta\left(a-bp\right)^2>0.$$

Therefore, $F_1(\alpha)$ is strictly positive for all values of α.

Appendix C. Proof of Theorem 2

For minimization, the total cost per unit time given by equation (57), it is required to calculate the first and second order derivatives of Eq. (57) with respect to α,

$$\frac{dTC}{d\alpha}=\frac{F_1'(\alpha)}{\sqrt{F_1(\alpha)}}\sqrt{\varphi_1}+F_2'(\alpha) \tag{C.1}$$

$$\frac{d^2TC}{d\alpha^2}=\frac{2F_1(\alpha)F_1''(\alpha)-\left(F_1'(\alpha)\right)^2}{2\left(\sqrt{F_1(\alpha)}\right)^3}\sqrt{\phi_1}+F_2''(\alpha)$$

$$\frac{d^2TC}{d\alpha^2}=\frac{2\left(\phi_2\phi_3-\phi_3^2\right)}{\left(\sqrt{F_1(\alpha)}\right)^3}\sqrt{\phi_1}. \tag{C.2}$$

Since $\varphi_2\varphi_3-\varphi_3{}^2=\frac{1}{4}\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right]c_s\eta(a-bp)^2>0$ and $F_1(\alpha)$ is positive (Theorem 3), $\frac{d^2TC}{d\alpha^2}>0$. So, $TC(\alpha)$ is convex. To obtain the necessary condition of minimization of $TC(\alpha)$, set the first order derivative $\frac{dTC}{d\alpha}$ is equal to zero and solve for α which is $\alpha=\frac{2T\varphi_3-\varphi_4}{2T\varphi_2}$.

Appendix D. Proof of Theorem 3

The optimal cycle length of the model with partial backlogging under full prepayment of the total purchase cost must be greater than the optimal cycle length of the model without any shortage under full advance payment policy. If $t_1=T$, then the model with partial backlogging becomes the model without any shortage and the corresponding total cost (from Eq. 10) is:

$$TC(T)=\frac{K}{T}+(1+I_eM)(1-r)c_i\left(a-bp\right)\left[1+\frac{(\theta+c)T}{2}\right]+\frac{1}{2}(c_h+\theta c_d)(a-bp)T. \tag{D.1}$$

For minimization of the total cost per unit time, it is necessary to calculate the first and second order derivatives of equation (7) with respect to T . These derivatives are:

$$\frac{dTC}{dT}=-\frac{K}{T^2}+\frac{1}{2}(1+I_eM)(1-r)c_i\left(a-bp\right)(\theta+c)+\frac{1}{2}(c_h+\theta c_d)(a-bp), \tag{D.2}$$

$$\frac{d^2TC}{dT^2}=\frac{2K}{T^3}. \tag{D.3}$$

To find the necessary condition of the minimization of the total cost per unit time, it is needed to set the first order derivative of TC with respect to the T equal to zero.

$$-\frac{K}{T^2}+\frac{1}{2}(1+I_eM)(1-r)c_i\left(a-bp\right)(\theta+c)+\frac{1}{2}(c_h+\theta c_d)(a-bp)=0 \tag{D.4}$$

Solving the above with respect to *T*, we have:

$$T=\sqrt{\frac{2K}{\left\{(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right\}\left(a-bp\right)}}. \tag{D.5}$$

Since $\frac{d^2TC}{dT^2}=\frac{2K}{T^3}>0$ the total cost per unit time TC has a global minimum value at

$$T^*=\sqrt{\frac{2K}{\left\{(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right\}\left(a-bp\right)}}. \tag{D.6}$$

Now the cycle length for the model without any shortage from Eq. (D.6) must be smaller than the cycle length in Eq. (19). So,

$$\sqrt{\frac{2K}{\left\{(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right\}\left(a-bp\right)}}<\sqrt{\frac{4\varphi_1\varphi_2-\varphi_4{}^2}{4(\varphi_2\varphi_3-\varphi_3{}^2)}}$$

$$\frac{2K}{\left\{(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right\}\left(a-bp\right)}<\frac{4K\frac{1}{2}\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d+c_s\eta)\right]\left(a-bp\right)}{\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right]c_s\eta\left(a-bp\right)^2}-\frac{\left\{(1+I_eM)(1-r)c_i-c_l\right\}^2(1-\eta)^2\left(a-bp\right)^2}{\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right]c_s\eta\left(a-bp\right)^2}$$

$$\frac{\left\{(1+I_eM)(1-r)c_i-c_l\right\}^2(1-\eta)^2}{\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right]c_s\eta}<\frac{2K\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d+c_s\eta)\right]}{\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right]c_s\eta\left(a-bp\right)}-\frac{2Kc_s\eta}{c_s\eta\left\{(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right\}\left(a-bp\right)}$$

$$\frac{\{(1+I_eM)(1-r)c_i-c_l\}^2(1-\eta)^2}{\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right]}<\frac{2K\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right]}{\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right](a-bp)}$$

$$(1-\eta)^2<\frac{2K\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right]}{(a-bp)\{(1+I_eM)(1-r)c_i-c_l\}^2}$$

$$\eta>1-\sqrt{\frac{2K\left[(1+I_eM)(1-r)c_i(\theta+c)+(c_h+\theta c_d)\right]}{(a-bp)\{(1+I_eM)(1-r)c_i-c_l\}^2}}$$. This completes the proof.

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