Stock replenishment policies for a vendor-managed inventory in a retailing system

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Abstract

Using vendor-managed inventory, the vendor determines the replenishment decisions at the location of buyers (retailers). This strategy is used primarily for handling demand fluctuations stemming from the Bullwhip effect, leading the system to prevent from holding excessive inventory that result in a reduction in the overall cost of the supply chain. The main advantages of VMI for vendors are higher levels of accessibility to inventory information and more direct contact with the customers. Similarly, VMI has some pros for the buyers, such as shared risk with upper levels of supply chain and reduction in their holding costs of inventory. In this paper, a vendor-managed inventory system is developed containing one vendor and two buyers in which the main assumption is that back-ordering and lost sales are permitted. In this system, (r, Q) and (R, T) replenishment policies are compared according to their performances to see which one performs more cost-efficiently when partial back-ordering is allowed. In accordance, mathematical models utilizing (r, Q) and (R, T) replenishment policies are developed, and algorithms for deriving the optimal replenishment decision variables are proposed. Moreover, significant differences between the two replenishment policies are discussed. The main finding obtained by this research is that when shortage is permitted, both (r, Q) and (R, T) replenishment policies under VMI have pros and cons in different contexts.

Keywords: Inventory management; vendor-managed inventory; partial back-ordering; periodic review; continuous review

1. Introduction

The most common inventory replenishment policies are the periodic-review and the continuous-review one. Under a periodic-review (R,T) policy, the inventory level is checked in constant time intervals of length T (Johari et al., 2018). If the inventory is at level y, a quantity $R - y$ is ordered to increase the inventory level to R. Under a continuous-review (r,Q) policy, a replenishment order for a quantity Q can be placed when the inventory level falls below a pre-specified level which is known as the reorder point r (Johansen & Hill, 2000).

Managing supply chain inventory is classified into centralized and decentralized (Petrovic et al., 1999). Centralized control will be implemented when a central decision-maker is responsible for determining the best policy to minimize the entire supply chain costs. Coordination and communication concepts need to be utilized to make a centralized supply chain network. Under decentralized control, supply chain members are responsible for their own local inventory decisions like replenishment decisions. Utilizing a centralized inventory management system may significantly result in dwindling the total inventory cost in the supply chain (Chen et al., 2016).

As it is known in traditional inventory systems, the optimal decision variables for buyers are not necessarily the optimized order for the vendor. Likewise, the optimal decision variables related to the vendor may increase the costs of the buyers. For overcoming these difficulties, the Vendor-Managed

Inventory (VMI) is introduced in the literature. VMI is a coordinated program which is cost benefited in the whole supply chain. The supplying organization under VMI policy, like a manufacturer or a supplier, takes full control of inventory management and is responsible for replenishment decisions for buyers or retailers (Rad et al., 2014). In other words, the responsibility of managing the customer's inventories is granted to the vendor (Kim & Shin, 2019). Moreover, the retailers provide the supplier with online inventory data, and they may set a service level for inventory stocks (Ryu et al., 2013). Many firms have implemented VMI programs to be more competitive, which leads to improving efficiency and the relationship between customer and supplier in a whole supply chain (Yao and Dresner, 2008).

Considering partial back-ordering (lost-sales and back-ordering simultaneously) is more pragmatic in firms when developing replenishment systems. According to the literature, there are not any studies for developing a replenishment inventory review system under a VMI policy with considering partial back-ordering while comparing the performance of (R, T) and (r, Q). Thus, (R, T) and (r, Q) replenishment reviews under VMI policy with partial back-ordering are proposed in this paper and compared to each other. In accordance, the primary targets of this research are twofold as follow:

- Taking into consideration partial back-ordering in the VMI model and the impact on the optimal decision variables.
- Comparing (R, T) and (r, Q) replenishment policies and determining the one which is more cost-efficient under different circumstances.

The rest of the paper is organized as follows. The literature review is provided in Section 2. Next, problem definition is explained in Section 3. Modeling materials are summarized in section 3.1. In Sections 3.2 and 3.3, mathematical models of (R, T) and (r, Q) with partial back-ordering under VMI are developed, respectively. Procedures and algorithms are presented in Section 4 to obtain the optimum value of decision variables. A numerical example borrowed from the literature is discussed in Section 5. Sensitivity analysis is provided in Section 6. Section 7 provides some managerial insights, and finally concluding remarks are presented in Section 8.

2. Literature Review

Cost reduction is one of the main objectives of the Vendor-Managed-Inventory approach. Benerjee (1986) proposed an integrated vendor-buyer inventory model, including one vendor, one buyer, and assumed that the production rate is finite. He concluded that the joint decision of ordering and price adjustment could be most economically beneficial for the vendor and the buyer in the supply chain. Waller and Johnson (1999) represented some advantages of VMI such as reduced cost and improved services. They clarified the role of each partner in the supply chain to achieve cost reductions and improved services. Achabal et al. (2000) signified the benefits of VMI for retailers and vendors. Improving the customer service level and more precise sales forecasting are some of the advantages for the retailer. Improving brand, preventing misleading data for production planning, and reducing incentives for gaming are advantages associated with a supplier in VMI. Goyal and Nebee (2000) studied the problem of single-vendor and single-buyer intending to determine economic production and shipment policy. Woo et al. (2001) presented an integrated inventory model comprising one vendor and multiple buyers. They stated that decreasing joint total cost among the vendor and buyers is obtained via decreasing the ordering cost. Pan and Yang (2001) studied the problem comprising one vendor and one buyer and proposed an integrated inventory model featuring controllable lead times. They modeled the problem by considering that the demand during the lead time has a normal distribution. Dong and Xu (2002) investigated the effects of VMI systems on supply chain effectiveness from both long and short term perspectives. Ouyang et al. (2004) developed the model of Pan and Yang (2001) assuming that the shortage is possible in lead time and lead time may be shortened with paying extra money. Huang and Yao (2005) investigated strategies of coordination in an inventory model with deteriorating items in the presence of one-vendor and multiple-buyers and suggested an algorithm for solving the model. They presented two models with normal stochastic lead time. Rusdiyansyah and Tsao (2005) studied the problem of the vendor's production scheduling and distribution issues under VMI by using routing and period traveling salesman problem.

Danese (2006) studied VMI both in upstream and downstream of the supply chain, aiming at coordinating the whole system. Chang et al. (2006) introduced an integrated inventory model including one vendor, one buyer with lead time, and suggested an interactive procedure to find the best solutions. They presented two models in which the ordering cost might depend on the lead time or not. They used the continuous review inventory system. Yao et al. (2007) suggested an analytical model aiming at studying the benefits of collaborative initiatives like VMI. They studied the impact of the main parameters of a supply chain on cost savings based on VMI policy. Dong et al. (2007) explored the conditions of adopting VMI such as market competitiveness, uncertainty with product demand, buyer operational costs, and buyer-supplier cooperation. Zhou and Wang (2007) presented a general production-inventory problem with one vendor and one buyer when shortages are permitted.

Yao et al. (2007) studied the VMI system by considering the assumptions stated in Dong and Xu (2002) and calculated that the replenishment quantities reduce in a supply chain under VMI. Partners of SC may implement this concept by establishing an electronic data interchange. Their modeling aims at obtaining the best investment and replenishment policies for the vendor and buyers. Siajadi et al. (2006) studied the SC model which includes one vendor but several buyers with multiple-shipment policy. Zhang et al. (2007) proposed a procedure to derive the best investment and replenishment approaches on VMI with the single-vendor multiple-buyers model. Claassen et al. (2008) studied the benefits of VMI and its success factors in the supply chain. They specified improved customer service, supply chain control, cost reduction and information sharing as VMI outcomes. Wong et al. (2009) coordinated a two-echelon supply chain employing a sales rebate contract under the VMI policy. They assumed the supply chain includes one supplier but several retailers.

Xu and Leung (2009) studied the retail channel under VMI policy and offered an analytical model to specify the inventory policy by considering the whole SC benefits. Zavanella and Zanoni (2009) presented an integrated model consisting of both production and inventory decisions in the presence of one vendor and multi-buyer with exploiting the consignment stock (CS) as a VMI policy. Razmi et al. (2010) compared the performance of both VMI and traditional system consideration of one-vendor and one-buyer. They concluded that VMI acts far more efficiently compared with the conventional system and imposes lower costs to a supply chain in all conditions. Guan and Zhao (2010) investigated the contracts for VMI based on the continuous review inventory system and developed a revenue-sharing contract.

Pasandideh et al. (2010) considered a VMI model containing single-supplier and single-retailer and modeled the problem for the economic order quantity (EOQ). They studied inventory management practices with and without implementing VMI and concluded that the system cost is lower when the shortages are backlogged. Chen and Chang (2010) proposed a problem comprising one vendor and multi-retailer for exponentially deteriorating items for obtaining the optimized retail price, replenishment cycle, and also the number of shipments, under the conditions of joint replenishment programs, vendor-buyer coordination and pricing policy. Hong and Yang (2010) studied the profitability of SC with and without implementing VMI using the economic order quantity (EOQ). Hoque (2011) developed two integrated inventory models in the problem, including one vendor and several buyers in order to obtain optimal production and shipment policies. Yu et al. (2012) presented a VMI system considering deteriorating raw materials like fruit, milk, and vegetables. In the proposed model, the vendor makes decisions related to inventories. They modeled the problem to find the optimal replenishment cycle for products and replenishment frequency for raw materials, to minimize inventory and deterioration costs. Cardenas-Barron et al. (2012) investigated a heuristic algorithm for deriving the decision variables of a VMI and EOQ considering multi-products and multi-constrained model in addition to linear and also constant backorder costs. It is assumed that the production flow will be synchronized shipping the lot equal or/and unequal sub-lots to minimize the total costs. Kang and Kim (2012) developed a nonlinear mixed-integer model with a single supplier and a central distribution center, multi-warehouses, and multi-retailers.

Xiao and Xu (2013) investigated the model with one-supplier and one-retailer with deteriorating items in order to find the optimal parameters related to the price and service levels under VMI policy. Sadeghi et al. (2013) presented a constrained model with several vendors, multi-retailer but just one warehouse under the assumption that space and numbers of orders are limited to the central warehouse.

The problem is formulated with integer nonlinear programming model and PSO, and genetic algorithms are exploited to find the order quantities. Tat et al. (2014) studied an EOQ based model for a single instantaneous deteriorating product under the VMI model in the SC containing two echelons. Later, they extended the previous model for non-instantaneous items (Tat et al., 2015).

Taleizadeh et al. (2015) considered the VMI model in a two-level supply chain comprising a single vendor and several retailers with deterministic and simultaneously price-sensitive demand. They specified the optimal price of retailers, the best value of replenishment for both raw materials and finished products to maximize the profit function under the Stackelberg approach. Mateen et al. (2015) investigated the model of single-vendor and multiple-retailers considering stochastic demands under VMI and developed approximate expressions for optimizing the cost functions of the whole system. Mateen and Chatterjee (2015) studied a VMI setting containing one vendor and multiple retailers under which used the VMI for coordinating the whole supply chain and reducing the cost of transportation.

Akbari Kaasgari et al. (2016) investigated a two-level supply chain with a perishable product including just one vendor but several retailers. They used two different meta-heuristic methods called Particle swarm (PSO) and Genetic algorithm (GA) and concluded that PSO works more efficiently for solving the proposed NP-hard model compared to the GA algorithm. Park et al. (2016) investigated a two-level supply chain comprising one manufacturer but several retailers under the VMI system. They utilized the genetic algorithm to optimize the routing problem and the profit of the supply chain when lost sales are allowed. Cai et al. (2017) studied a two-level supply chain that sells two different products with two different kinds of brands with uncertain demand. They used the VMI model to investigate the effect of substitutability of products on the proposed supply chain. Liu et al. (2017) considered the VMI approach for integrating conventional and online commerce channels in the military context for improving the power of bargaining among military suppliers. Han et al. (2017) utilized a tri-level model for coordinating the inventory decisions between a vendor and a buyer in a three-level supply chain. Filho et al. (2018) investigated the application of the VMI system for predicting the value of demand in the sector of animal industry by applying customer relationship management (CRM). Hu et al. (2018) took into account a supply chain coordination method and vendor managed inventory along with fairness concern to increase the profit of suppliers to prevent the bankruptcy of the supplier's because of low-profit margins. Sainathan and Groenevelt (2018) used five different supply chain contracts (sales rebate, quantity discount, revenue sharing, buyback, and quantity flexibility) to coordinate a supply chain containing one vendor and one retailer using the VMI approach and newsvendor model.

Chen (2018) studied a supply chain with a perishable product that must decide on inventory and production decisions under the VMI model. They used pricing decisions and promotion effort to coordinate the supply chain and deduced that just the strategies in which tries to dwindle the cost of transportation and increase the competitiveness of supply chain members, can drastically create a win-win situation for all SC members. Weraikat et al. (2019) showed that by using the VMI model in a pharmaceutical supply chain, hospitals might be able to reduce the number of excessive drugs that may be on the verge of expiration. A critical challenge for vendors and buyers is that they are willing to hold the least possible value of the inventory. Bieniek (2019) considered the VMI policy along with the consignment (VMCI) with uncertain demand function. They utilized a two-stage model where first, the vendor specified the order quantity and price of the product to maximize the vendor's profit, and second, the retailer determined the retail price.

Gharaei et al. (2019) developed a mathematical VMI model containing several buyers and several products under stochastic restraints. They also take into account green policies and quality control considerations for determining the optimal size of the batch. Bai et al. (2019) considered a two-echelon supply chain with a perishable product consisting of one manufacturer and two retailers under VMI policy. Besides, they used a cap-and-trade policy to curtail the amount of emission and profitability in the decentralized scenario. Stellingwerf et al. (2019) used the VMI model as a cooperative approach for reducing the economic and environmental effects of carbon dioxide emissions on the supply chain. They used Shapley value and cooperative games to obtain a fair allocation of costs and benefits among supply chain members. Golpira (2020), utilized the expert system to design a novel MILP to solve an integrated VMI model into a facility location problem in a constructional supply chain and inferred that the higher amounts of replenishment frequency could lead to lower costs of inventory and more advantages to the supply chain members.

As mentioned in the previous section, to our best knowledge, there are no studies that evaluate the VMI model with considering partial back-ordering while comparing two different review strategies ((r, Q) and (R, T)) to see which one performs more cost-efficient under different circumstances. To fill this research gap, a mathematical model utilizing (r, Q) and (R, T) policies under VMI is provided in this research to assess the profitability of each policy.

3. Problem definition

As stated, VMI helps the supply chain to be coordinated and more cost-efficient compared to the traditional policies like retailer managed inventory (RMI) systems. This study aims to improve coordination between vendor and buyers under VMI policy when partial back-ordering is permitted. According to the competitive environments, mechanisms such as coordination, which leads to enhanced responsiveness and cost-efficiency of the whole chain, can be advantageous. Besides, it is more pragmatic to consider lost-sales and backorders for developing inventory models in real cases. In this paper, the problem is to analyze VMI systems for one vendor and two buyers under periodic review (R, T) and continuous review (r, Q) replenishment policies with partial back-ordering. Another purpose of this study is comparing two proposed replenishment systems and discussing their significant differences. Because of the mathematical complexities in obtaining optimal decision variables, there is a need to develop an efficient procedure for deriving them in both of the replenishment review systems. For this purpose, after modeling the problem, two algorithms are considered to obtain optimal decisions.

In consequence, we are going to optimize a single-vendor with two buyers model under VMI policy and make a comparison between the two inventory replenishment policies ((r, Q) and (R, T)). As mentioned in Section 1, the vendor is responsible for providing raw materials, producing items, and supplying items to the two buyers under the VMI policy. So, a two-echelon SC is under consideration, containing a vendor but two buyers. The following assumptions are considered in this study:

- The model has a single vendor but two buyers.
- The vendor supplies only one item for buyers.
- Buyers share their information related to replenishment decision parameters to the vendor.
- The inventory can be reviewed with (r, Q) or (R, T) policies.
- The rate of production is finite and higher than the demand rate.
- The lead time depends on the lot sizes and delay times and is common for the buyers.
- Transportation-related processes can result in fixed delay times.
- Two machines are available at the vendor's site to produce items.
- The cycle time of production is invariant and the same for each buyer.
- The vendor delivers the goods to buyers when the whole amount of production is ready.
- The common delivery cycle will be used to ship items to two buyers.

3.1. Modeling

Following parameters and variables are defined in this section to develop mathematical models:

Notations:

D_i	Average demand rate for buyer i ($D = \sum_i D_i$).
P	Production rate at the vendor's site, which is higher than the total demand rates.
Q_i	The order quantity for buyer i.
A_s	Cost of per order paid by the vendor.
A_i	Cost of per order paid by buyer i.
F_i	Transportation cost per shipment paid by buyer i.
w	The weighting factor for the vendor's cost of the order.
h_s	The unit cost of holding per time for the vendor.
h_{Bi}	The unit cost of holding per time paid by buyer i.
π_i	Unit penalty cost for back-ordering paid by buyer i.
π'_i	The unit cost of lost-sale paid by buyer i.
$\bar{b}_i$	Backlogged value of buyer i.

$\bar{b}$	The average amount of back-ordering in the supply chain.
b_i	Constant delay time for transporting items from the vendor to buyer *i*.
S_i	Safety stock of buyer *i*.
S	Average safety stocks in a chain.
R	Maximum inventory level under (*R*, *T*) system.
r	Point of reordering under (*r, Q*) replenishment review.
T	Common cycle time
z_p	Inverse of service level distribution (normal distribution).
μ	The expected value of demand throughout lead time.
σ	The standard deviation of demand throughout the lead time.
$l(Q)$	Buyer's common lead time under (*r, Q*).
$l(DT)+T$	Buyer's common lead time under (*R*, *T*).
$1\text{-}\alpha$	A fraction of the shortage that will be lost.
KB_i^{rQ}	Total cost paid by buyer *i* under VMI with (*r, Q*) system.
KB_i^{RT}	Total cost paid by buyer *i* under VMI with (*R*, *T*) system.
KS^{rQ}	Total cost paid by the vendor under VMI with (*r, Q*) system.
KS^{RT}	Total cost paid by the vendor under VMI with (*R*, *T*) system.
TC_{BACK}^{rQ}	The Cost of backorders under VMI in (*r, Q*).
TC_{BACK}^{RT}	The Cost of backorders under VMI in (*R*, *T*).
TC_{LOST}^{rQ}	The Cost of lost-sales under VMI in (*r, Q*).
TC_{LOST}^{RT}	The Cost of lost-sales under VMI in (*R, T*).
TC^{rQ}	The Total cost of SC under the VMI model with (*r, Q*) system.
TC^{RT}	The total cost of SC under the VMI model with (*R*, *T*) system.

It is assumed that the cycle of production time is equal for each buyer. Under this assumption, the vendor orders raw materials for both buyers simultaneously under the VMI policy and pays one ordering cost instead of two different orders for the buyers. As well, the reduction in the cost of ordering is one of the benefits of VMI in comparison to traditional retailer-managed inventory (RMI) systems, since the vendor has sufficient information about the retailers' inventory levels [Chang et al., 2006, Woo et al., 2001, Zhang et al., 2007]. Hence, it is rational to apply methods such as the weighting factor method (Rad et al., 2014) to enhance the cost efficiency of the whole chain as a result of ordering cost reduction in VMI systems. In this paper, the weighting factor technique in Rad et al. (2014) is utilized to reduce the total ordering costs. Using weighting factors in developing VMI systems is beneficial, and thereby lowers ordering costs, resulting in a reduction in the total cost of chain.

As discussed by Rad et al. (2014), ordering cost for the vendor (A_s) will consist of two elements: Firstly, the ordering cost of raw materials or work-in-process and secondly, the cost of setting up for producing products for each buyer. As an example in Rad et al. (2014), suppose that 30% of the cost of ordering cost is related to raw materials, and 70% is allocated for setting up production equipment. Therefore, the cost of ordering paid by the vendor is total of below items:

- The cost of ordering the raw materials for buyers is equal to $0.3A_s$.
- Setting-up cost for the first buyer is equal to $0.7A_s$.
- Setting-up cost for the second buyer is equal to $0.7A_s$.

Thus, the total cost of ordering paid by the vendor is equal to $(0.3+0.7+0.7)A_s = 1.7A_s$. So, the weighting factor for ordering the cost of the vendor is equal to $w=1.7$. Before calculating the cost of SC under the VMI system, it is worthy to calculate ordering cost with the weighting factor technique for the vendor's cost of ordering. The vendor decides when and how much to order and produces $Q=Q_1+Q_2$ (there are two-buyers), and then sends the prepared items to buyers. Furthermore, there are two machines at the vendor's site in order to respond to the buyers' demands. Whenever each machine has done the production process for one of the buyers (for instance, Q_1), the vendor waits for the other machine to complete its production process (for instance, Q_2). Later, the vendor ships the whole lot size $Q=Q_1+Q_2$ to the buyers to meet the buyers' demands ($D = D_1 + D_2$). In other words, the problem is to produce and ship the whole lot size ($Q=Q_1+Q_2$) in common cycle time for both of the buyers and then ship the finished goods to buyers in common delivery time. It is noteworthy to say that the cycle times of buyers are the same.

It is worth noting that each member of the chain makes decisions separately in RMI systems, which may not be optimal for other members, but decisions in VMI systems are centralized by which the whole chain could be in profit. Additionally, in traditional RMI systems, the same production cycle time cannot be used because decisions are neither centralized nor in the control of the vendor. However, in VMI systems, the vendor may utilize the same production cycle time, which is superior to the RMI systems (Rad et al., 2014). Another significant point is that there are two different buyers with different cost parameters. Hence, these two buyers should have different order frequencies. In this paper, it is assumed they have the same order frequency known as common cycle time as an advantage of VMI systems. Therefore, to avoid any yielding higher system-wide cost, a common delivery cycle (lead-time) is exploited to develop the models, which may reduce the whole VMI system's cost. As we have mentioned, the cycle of production time (*T*) is identical for each buyer. Besides, two machines are available at the site of the vendor to produce the received orders. Afterward, the vendor ships the whole lot size (*D*) to the buyers, which require a common cycle and delivery time.

According to the above explanations, inventory costs will be moved to the vendor (Rad et al., 2014; Yao et al., 2007; Pasandideh et al., 2010) under VMI, the total cost of the order would be $\frac{D}{Q}(F_1+F_2+A_1+A_2)+w\frac{D}{Q}A_s=\frac{F_1+F_2+A_1+A_2+wA_s}{T}$ which belongs to the vendor. As a consequence, the derived equation is used as total ordering costs, wherever needed in (*r*, *Q*) or (*R*, *T*) policies.

3.2. Modeling (R, T) with partial back-ordering

The aggregate cost of VMI consists of the cost of the supplier and buyers. Additionally, the inventory costs of buyers moved to the vendor (Rad et al., 2014; Yao et al., 2007; Pasandideh et al., 2010), and the total cost of the buyers is equal to zero under VMI ($KB_1^{RT}=KB_2^{RT}=0$). Thus, the total cost of SC under the VMI system is $TC^{RT}=KB_1^{RT}+KB_2^{RT}+KS^{RT}=KS^{RT}$, meaning that the VMI cost is the same as the cost of the vendor. For that reason, the vendor is imposed by the cost of ordering, cost of holding, costs of back-ordering and costs of lost-sales.

Note that, as it was mentioned in section 2.1, the total cost of ordering with considering the weighting factor is equal to $\frac{D}{Q}(F_1+F_2+A_1+A_2)+w\frac{D}{Q}A_s=\frac{F_1+F_2+A_1+A_2+wA_s}{T}$ .The total cost of the vendor as a supplier is dependent on the lost-sales rate. Accordingly, the back-ordering case and lost-sales should be evaluated before calculating the total cost. Thus, we have:

$$TC_{BACK}^{RT}=\frac{F_1+F_2+A_1+A_2+wA_s}{T}+h_{B1}\left(\frac{D_1T}{2}+S_1\right)+h_{B2}\left(\frac{D_2T}{2}+S_2\right)+\frac{h_sD_1T}{2}\left(1-\frac{D_1}{P}\right)+\frac{h_sD_2T}{2}\left(1-\frac{D_2}{P}\right)+\frac{\pi_1}{T}\bar{b}_1\left(R_1,l\left(DT\right)+T\right)+\frac{\pi_2}{T}\bar{b}_2\left(R_2,l\left(DT\right)+T\right) \quad (1)$$

$$TC_{LOST}^{RT}=\frac{F_1+F_2+A_1+A_2+wA_s}{T}+h_{B1}\left(\frac{D_1T}{2}+S_1+\bar{b}_1\left(R_1,l\left(DT\right)+T\right)\right)+h_{B2}\left(\frac{D_2T}{2}+S_2+\bar{b}_2\left(R_2,l\left(DT\right)+T\right)\right)+\frac{h_sD_1T}{2}\left(1-\frac{D_1}{P}\right)+\frac{h_sD_2T}{2}\left(1-\frac{D_2}{P}\right)+\frac{\pi_1^{'}}{T}\bar{b}_1\left(R_1,l\left(DT\right)+T\right)+\frac{\pi_2^{'}}{T}\bar{b}_2\left(R_2,l\left(DT\right)+T\right) \quad (2)$$

Equation (1) is the total cost of the chain by considering the fully back-ordering case when lost-sale is not involved. Associated costs in the formulation of Equation (1) include ordering cost, holding costs for both buyers and vendor, and back-ordering cost for both of the buyers. Equation (2) is the total cost of the chain, in which the lost-sales are taken into account. Related costs include costs of the order, costs of holding, and cost of lost-sales. Note that weighting factor, described in Section 2.1, is used in the model formulations (i.e., Equations (1) and (2) to reduce the ordering costs as it is shown that would be more profitable) when developing VMI systems (Rad et al., 2014, Yao et al., 2007).

As discussed in (Rad et al., 2014), since the vendor provides the same item for each buyer, the cost of holding is equal for the buyers. In accordance, the average holding costs are used for calculating the safety stocks holding cost. Likewise, backorders with the penalty costs of π_1 and π_2 and lost sales with the penalty costs $\pi_1^{'}$ and $\pi_2^{'}$ are the same for both of the buyers. Thus, we can use the average of them for calculating the total cost of backorders and lost sales. Therefore, we have:

$$TC_{BACK}^{RT}=\frac{F_1+F_2+A_1+A_2+wA_s}{T}+h_{B1}\left(\frac{D_1T}{2}\right)+h_{B2}\left(\frac{D_2T}{2}\right)+\frac{\left(h_{B1}+h_{B2}\right)}{2}S+\frac{h_sDT}{2}\left(1-D/2P\right)+\frac{\left(\pi_1+\pi_2\right)}{2T}\bar{b}\left(R,l\left(DT\right)+T\right) \quad (1.a)$$

$$TC_{LOST}^{RT}=\frac{F_1+F_2+A_1+A_2+wA_s}{T}+h_{B1}\left(\frac{D_1T}{2}\right)+h_{B2}\left(\frac{D_2T}{2}\right)+\frac{(h_{B1}+h_{B2})}{2}S+\frac{(h_{B1}+h_{B2})}{2}\bar{b}\left(R,l\left(DT\right)+T\right)$$
$$+\frac{h_sDT}{2}(1-D/2P)+\frac{(\pi_1^{'}+\pi_2^{'})}{2T}\bar{b}\left(R,l\left(DT\right)+T\right) \quad (2.a)$$

It is presumed that *α* percent of orders are back-ordered, and *(1-α)* percent are lost sales. Besides, in the periodic review inventory management, the safety stock is equal to $R-\mu_{L+T}$, where R is the maximum level of inventory and μ_{L+T} is the average quantity of demand during lead-time.

When there is partial back-ordering, SC total cost is $TC^{RT}=\alpha TC_{BACK}^{RT}+(1-\alpha)TC_{LOST}^{RT}$ where α is the backorder rate and (1-α) is the rate at which sales lost. Hence, the total cost of SC under VMI is reformulated with Equation (3):

$$TC^{RT}=\frac{F_1+F_2+A_1+A_2+wA_s}{T}+\left(\frac{h_{B1}D_1+h_{B2}D_2}{2}\right)T+\frac{(h_{B1}+h_{B2})}{2}\left(R-\mu_{L+T}\right)+\frac{h_sDT}{2}(1-D/2P)$$
$$+(1-\alpha)\frac{(h_{B1}+h_{B2})}{2}\bar{b}\left(R,l\left(DT\right)+T\right)+\left(\alpha\left(\pi_1+\pi_2\right)+(1-\alpha)(\pi_1^{'}+\pi_2^{'})\right)\frac{\bar{b}\left(R,l\left(DT\right)+T\right)}{2T} \quad (3)$$

Equation (3) is the total cost of the chain with partial back-ordering. Decision variables (i.e., cycle time (T) and maximum level of inventory (R)) will be derived through the proposed solution procedure described in Section 3.1.

3.3. Modeling (r, Q) with partial back-ordering

As stated in the previous section, under VMI policy all the replenishment decisions and inventory cost for buyers moved to the vendor. Therefore, the total cost of inventories for buyers is equal to zero ($KB_1^{rQ}=KB_2^{rQ}=0$) (see in Rad et al., 2014; Yao et al., 2007; Pasandideh et al., 2010) and the total cost of SC is $TC^{rQ}=KB_1^{rQ}+KB_2^{rQ}+KS^{rQ}=KS^{rQ}$. Since the total cost of the vendor is dependent on back-ordered and lost-sales rate, they are formulated separately as Equations (4) and Equation (5):

$$TC_{BACK}^{rQ}=\frac{F_1+F_2+A_1+A_2+wA_s}{T}+h_{B1}\left(\frac{Q_1}{2}+S_1\right)+h_{B2}\left(\frac{Q_2}{2}+S_2\right)+\frac{h_sQ_1}{2}\left(1-\frac{D_1}{P}\right)$$
$$+\frac{h_sQ_2}{2}\left(1-\frac{D_2}{P}\right)+\pi_1\frac{D_1}{Q_1}\bar{b}_1(r_1)+\pi_2\frac{D_2}{Q_2}\bar{b}_2(r_2) \quad (4)$$

$$TC_{LOST}^{rQ} = \frac{F_1+F_2+A_1+A_2+wA_s}{T} + h_{B1}\left(\frac{Q_1}{2}+S_1+\bar{b}_1(r_1)\right) + h_{B2}\left(\frac{Q_2}{2}+S_2+\bar{b}_2(r_2)\right) + \frac{h_s Q_1}{2}\left(1-\frac{D_1}{P}\right) + \frac{h_s Q_2}{2}\left(1-\frac{D_2}{P}\right) + \pi_1^{'}\frac{D_1}{Q_1}\bar{b}_1(r_1) + \pi_2^{'}\frac{D_2}{Q_2}\bar{b}_2(r_2) \tag{5}$$

Equation (4) is related to the total cost of the chain when there is a back-ordered case and comprising ordering costs, holding costs, and back-ordering costs, respectively. Moreover Equation (5) is the formulation of the total cost by considering the lost-sales case and terms are associated with ordering costs, holding costs, and lost-sales costs. Noteworthy to mention that the weighting factor is exploited to calculate ordering cost, which may decrease the total cost of SC.

Since the total cost of the VMI is equal to $TC^{rQ} = \alpha TC_{BACK}^{rQ} + (1-\alpha) TC_{LOST}^{rQ}$, where α is the back-ordering rate while (1-α) is the rate at which sales lost, the total cost of the SC is formulated as Equation (6):

$$TC^{rQ} = \frac{F_1+F_2+A_1+A_2+wA_s}{T} + h_{B1}\left(\frac{Q_1}{2}+S_1\right) + h_{B2}\left(\frac{Q_2}{2}+S_2\right) + (1-\alpha)h_{B1}\bar{b}_1(r_1) + (1-\alpha)h_{B2}\bar{b}_2(r_2) + \frac{h_s Q_1}{2}\left(1-\frac{D_1}{P}\right) + \frac{h_s Q_2}{2}\left(1-\frac{D_2}{P}\right) + \left(\alpha\pi_1 + (1-\alpha)\pi_1^{'}\right)\frac{D_1}{Q_1}\bar{b}_1(r_1) + \left(\alpha\pi_2 + (1-\alpha)\pi_2^{'}\right)\frac{D_2}{Q_2}\bar{b}_2(r_2) \tag{6}$$

As described earlier in section 2.2, as the vendor sells identical goods to each buyer with the same cycle of production time, their holding costs are the same. Consequently, the average holding costs are utilized to calculate the holding cost of safety stocks. Moreover, the cost of backorders (π_1 and π_2) and lost sales ($\pi_1^{'}$ and $\pi_2^{'}$) are the same, and the average of them is used for calculating the total cost of backorders and lost sales. So, the SC total costs reformulated via Equation (7):

$$TC^{rQ} = \frac{F_1+F_2+A_1+A_2+wA_s}{T} + \frac{1}{2}D_1 T h_{B1} + \frac{1}{2}D_2 T h_{B2} + \frac{1}{2}(h_{B1}+h_{B2})SS + \frac{1}{2}h_s DT(1-D/2P) + (1-\alpha)\frac{(h_{B1}+h_{B2})}{2}\bar{b}(r, l(Q)) + \frac{\alpha\pi_1 + (1-\alpha)\pi_1^{'} + \alpha\pi_2 + (1-\alpha)\pi_2^{'}}{2T}\bar{b}(r, l(Q)) \tag{7}$$

As it is apparent, in (r, Q) system, safety stock is equal to $r-\mu_L$, which r is the reordering parameter, and μ_L is the expected value of demand throughout the lead-time. As a result, we can swap safety stock with r-μ_L wherever needed in Equation (7) to reformulate it. Therefore, the total cost of SC is formulated by Equation (8):

$$TC^{rQ} = \frac{F_1 + F_2 + A_1 + A_2 + wA_s}{T} + \frac{1}{2}D_1Th_{B1} + \frac{1}{2}D_2Th_{B2} + \frac{1}{2}\left(h_{B1} + h_{B2}\right)\left(r - \mu_L\right) + \frac{1}{2}h_sDT\left(1 - D/2P\right)$$
$$+ (1-\alpha)\frac{(h_{B1} + h_{B2})}{2}\bar{b}\left(r, l\left(Q\right)\right) + \frac{\alpha\pi_1 + \left(1-\alpha\right)\pi_1^{'} + \alpha\pi_2 + \left(1-\alpha\right)\pi_2^{'}}{2T}\bar{b}\left(r, l\left(Q\right)\right) \quad (8)$$

Decision variables involved in the modeling can be obtained through the solution procedure presented in Section (3.2).

4. Solution Methodology

To find the decision variables for both (*R*, *T*) and (*r*, *Q*) model with partial back-ordering, the derivation technique is exploited. Subsequently, two separate algorithms are introduced in this section to reach the optimal decision variables for both of the replenishment review systems. Thus, the decision variables are derived in the next two sections.

4.1. Deriving (R, T) decision variables

For finding the variable *R* in (*R*, *T*) replenishment system, we derive from equation (3) with respect to *R* and put it equal to zero. Moreover, we know $\frac{\partial\bar{b}\left(R, l\left(DT\right) + T\right)}{\partial R} = -\bar{F}\left(R\right)$ (see Appendix A). Therefore, we have:

$$\frac{\partial TC^{RT}}{\partial R} = 0 \rightarrow \frac{\partial TC^{RT}}{\partial R} = \frac{\left(h_{B1} + h_{B2}\right)}{2} - [\left(1-\alpha\right)\frac{\left(h_{B1} + h_{B2}\right)}{2} + \alpha\frac{\left(\pi_1 + \pi_2\right)}{2T} + \left(1-\alpha\right)\frac{\left(\pi_1^{'} + \pi_2^{'}\right)}{2T}]\bar{F}\left(R\right) = 0 \quad (9)$$

Hence, the cumulative distribution of *R* is calculated via Equation (10):

$$\bar{F}\left(R\right) = \frac{\left(h_{B1} + h_{B2}\right)T}{T\left(1-\alpha\right)\left(h_{B1} + h_{B2}\right) + \alpha\left(\pi_1 + \pi_2\right) + \left(1-\alpha\right)\left(\pi_1^{'} + \pi_2^{'}\right)} \quad (10)$$

Moreover, we assume that demand during the lead time follows a normal distribution below $x \sim N\left(D\left(l\left(DT\right) + T\right), \sigma^2\left(l\left(DT\right) + T\right)\right)$. So, we have:

$$\bar{b}\left(R, l\left(DT\right) + T\right) = \int_R^\infty \left(x - R\right) f\left(x\right) dx = \int_R^\infty \left(x - R\right)\frac{1}{\sqrt{2\pi}\sigma\sqrt{l\left(DT\right) + T}} e^{-\frac{1}{2}\left(\frac{x-R}{\sigma\sqrt{l\left(DT\right)+T}}\right)^2} dx.$$

Note that the vendor produces $Q=Q_1+Q_2$ and then ships them to the buyers. Additionally, there are two available machines at the vendor's site, each one has the production rate P, and machines are responsible for producing items for buyers and meeting buyers' demands (D_1+D_2). As the vendor waits for the machines to produce items and then ships the whole lot size for buyers, the maximum fixed amount of delay as a result of waiting, quality control, and moving is equal to the maximum delay times $Max\{b_1, b_2\}$. As a result, the common procurement lead time is $l\left(DT+T\right)=\frac{DT}{2P}+\max\left(b_1,b_2\right)+T$ under (R,T) system. As we have $l\left(DT+T\right)=\frac{DT}{2P}+\max\left(b_1,b_2\right)+T$, hence $\bar{b}\left(R,l\left(DT\right)+T\right)=\sigma\sqrt{(DT/2P)+\max\left(b_1,b_2\right)+T}\,L^{'}(u)$ (See Appendix B). Besides, the standard deviation during the lead time is $\sigma_l=\sqrt{\sigma_1^2+\sigma_2^2}\sqrt{(DT/2P)+\max\left(b_1,b_2\right)+T}$, and $SS=z_p\sigma_{L+T}$. Consequently, the value of safety stock is $SS=z_p\sqrt{\sigma_1^2+\sigma_2^2}\sqrt{(DT/2P)+\max\left(b_1,b_2\right)+T}$. Furthermore, the average back-ordering could be calculated with the equation $\bar{b}\left(R,l\left(DT\right)+T\right)=\sqrt{\sigma_1^2+\sigma_2^2}\sqrt{(DT/2P)+\max\left(b_1,b_2\right)+T}\,L^{'}(u)$. Now, the cost function of SC can be rewritten by inserting the above equations wherever needed in Equation (3). According to the above-provided material, the Equation (11) will be obtained using Equation (3):

$$\begin{aligned}TC^{RT}=&\frac{F_1+F_2+A_1+A_2+wA_s}{T}+h_{B1}\left(\frac{D_1T}{2}\right)+h_{B2}\left(\frac{D_2T}{2}\right)+\frac{\left(h_{B1}+h_{B2}\right)}{2}z_p\sqrt{\sigma_1^2+\sigma_2^2}\sqrt{\frac{DT}{2P}+\max\left(b_1,b_2\right)+T}\\&+\frac{h_sDT}{2}\left(1-\frac{D}{2P}\right)+\left(1-\alpha\right)\frac{\left(h_{B1}+h_{B2}\right)}{2}\sqrt{\sigma_1^2+\sigma_2^2}\sqrt{\frac{DT}{2P}+\max\left(b_1,b_2\right)+T}\,L^{'}(u)\\&+\frac{\sqrt{\sigma_1^2+\sigma_2^2}}{2T}\sqrt{DT/2P+\max\left(b_1,b_2\right)+T}\,L^{'}(u)\left(\alpha\left(\pi_1+\pi_2\right)+\left(1-\alpha\right)\left(\pi_1^{'}+\pi_2^{'}\right)\right)\end{aligned}\tag{11}$$

To derive the optimum value for common cycle time (T), Equation (11) is derived with respect to T and Equation (12) is obtained:

$$\frac{\partial TC^{RT}}{\partial T}=-\left(\frac{F_1+F_2+A_1+A_2+wA_s}{T^2}\right)+\frac{h_{B1}D_1}{2}+\frac{h_{B2}D_2}{2}+\frac{\left(h_{B1}+h_{B2}\right)}{2}z_p\sqrt{\sigma_1^2+\sigma_2^2}\,\frac{\frac{D}{2P}+1}{2\sqrt{(DT/2P)+\max\left(b_1,b_2\right)+T}}$$

$$+\frac{h_sD}{2}\left(1-\frac{D}{2P}\right)+\left(1-\alpha\right)\frac{\left(h_{B1}+h_{B2}\right)}{2}L'(u)\sqrt{\sigma_1^2+\sigma_2^2}\,\frac{\frac{D}{2p}+1}{2\sqrt{(DT/2P)+\max\left(b_1,b_2\right)+T}} \tag{12}$$

$$-\left(\frac{\alpha\left(\pi_1+\pi_2\right)+(1-\alpha)(\pi_1'+\pi_2')}{2T^2}\right)L'(u)\sqrt{\sigma_1^2+\sigma_2^2}\left(\sqrt{(DT/2P)+\max\left(b_1,b_2\right)+T}-\frac{\frac{D}{2P}+1}{4T\sqrt{(DT/2P)+\max\left(b_1,b_2\right)+T}}\right.$$

With setting the Equation (12) equal to zero, the optimal quantity of common cycle time is worked out by Equation (13):

$$T_{RT}^{*}=\sqrt{\frac{1}{2}\times\frac{2\left(F_1+F_2+A_1+A_2+wA_s\right)+\left[\alpha\left(\pi_1+\pi_2\right)+\left(1-\alpha\right)\left(\pi_1'+\pi_2'\right)\right]\times L'(u)\sqrt{\sigma_1^2+\sigma_2^2}\sqrt{(DT/2P)+\max\left(b_1,b_2\right)+T}}{\left(\begin{array}{l}\frac{h_{B1}D_1}{2}+\frac{h_{B2}D_2}{2}+\frac{h_sD}{2}\left(1-\frac{D}{2P}\right)+\frac{\left(h_{B1}+h_{B2}\right)}{2}z_p\sqrt{\sigma_1^2+\sigma_2^2}\,\frac{\frac{D}{2P}+1}{2\sqrt{(DT/2P)+\max\left(b_1,b_2\right)+T}}\\+\left[\frac{\left(1-\alpha\right)\left(h_{B1}+h_{B2}\right)}{2}+\frac{\alpha\left(\pi_1+\pi_2\right)+\left(1-\alpha\right)\left(\pi_1'+\pi_2'\right)}{2T}\right]L'(u)\sqrt{\sigma_1^2+\sigma_2^2}\,\frac{\frac{D}{2P}+1}{2\sqrt{(DT/2P)+\max\left(b_1,b_2\right)+T}}\end{array}\right)}} \tag{13}$$

As can be seen in Equation (13), T is a function over itself, and it is not workable to calculate the common cycle time (T) directly. That is why Equation (13) is a recursive equation and there is a need to develop an efficient algorithm to determine T. For overcoming this difficulty and deriving T, the following algorithm is developed to find the optimized quantity of T and the maximum level inventory (R) as the critical decision variables in (R, T) replenishment review system:

Step1. Calculate the initial value of T using $T_1=\sqrt{\frac{2\left(F_1+F_2+A_1+A_2+wA_s\right)}{h_{B1}D_1+h_{B2}D_2+h_sD\left(1-D/2P\right)}}$

Step2. Calculate $\overline{F}\left(R\right)$ and find the R using Equation (10).

Step3. Calculate the $L'(u)$ as the right-hand unit of normal linear-loss integral.

Step4. Calculate T_2 by using Equation (13) with replacing of T_1 in the equation.

Step5. If | T_2-T_1|=0,

Denote T_2 by T_{RT}^*

Calculate TC^{RT} using Equation (11).

Else

Set $T_2 \rightarrow T_1$

Go to Step 2.

Step6. Calculating $Q_1 = D_1 T_{RT}^*$ and $Q_2 = D_2 T_{RT}^*$.

Step7. Stop.

The algorithm starts with an initial T denoted by T_1 (step 1). T_1 is the initial value of period length when there is not any backlogging in the inventory system, which is appropriate for the algorithm to be started with. For the current common cycle time (T_1), the decision variable R can be calculated using Equation (10) in step 2. Step 3 declares calculating $L^{'}(u)$ as of normal integral. According to supplementary materials presented in Appendix B, one can calculate the parameter u using $u = \dfrac{x - R}{\sigma\sqrt{(DT/2P) + \max\{b_1, b_2\} + T}}$. Note that the current T will be used in steps 2 and 3 for any calculation. Step 4 updates the common cycle time using Equation (13). The updated common cycle time will be known as the optimum value if the updated common cycle time and the current cycle time, for instance, in T_2 and T_1 in the first iteration, becomes equal. In other words, the algorithm continues until coverage will occur for the period length. Otherwise, the algorithms go to steps 2 and 3, to calculate the new R and $L^{'}(u)$ for the updated period length. This procedure continues to obtain the best quantity for the length of the period. After finishing the algorithm and finding optimum value for common cycle time T_{RT}^*, the total cost of the chain can be calculated with Equation (10). Finally, the optimum order quantities for the buyers will be determined in step 6.

4.2. Deriving (r, Q) decision variables

Since we know $\frac{\partial \bar{b}(r)}{\partial r} = -F(r)$ (see Appendix A), for finding the optimal r as reorder point Equation (8) is obtained:

$$\frac{\partial TC^{rQ}}{\partial r} = 0 \rightarrow \frac{1}{2}\left(h_{B1}+h_{B2}\right) - \frac{\alpha\pi_1 + (1-\alpha)\pi_1^{'} + \alpha\pi_2 + (1-\alpha)\pi_2^{'}}{2T}\bar{F}(r) = 0 \tag{14}$$

$$\bar{F}(r) = \frac{\left(h_{B1}+h_{B2}\right)T}{\alpha\left(\pi_1+\pi_2\right) + (1-\alpha)\left(\pi_1^{'}+\pi_2^{'}\right) + (1-\alpha)\left(h_{B1}+h_{B2}\right)T} \tag{15}$$

Therefore, the cumulative distribution of r (reorder point) is defined via Equation (15).

As discussed in Section 3.1, the vendor produces $Q=Q_1+Q_2$ and then ships them to the buyers with the two available machines. After producing the total lot size, the vendor ships them to the buyers. Thus, the common procurement lead time is $l(Q) = \frac{Q}{2P} + \max(b_1, b_2)$ under (r, Q). As well, in the proposed (r, Q) model, the delivery time can be calculated with $l(Q) = Q/2P + \max(b_1,b_2) = DT/2P + \max(b_1,b_2)$ $\bar{b}(r) = \sigma\sqrt{DT/2P + \max(b_1,b_2)}L^{'}(u)$. Safety stock in the current replenishment review system can be calculated with $SS = z_p\sigma_L = k\sqrt{\sigma_1^2+\sigma_2^2}\sqrt{DT/2P+\max(b_1,b_2)}$ and $\sigma = \sqrt{\sigma_1^2+\sigma_2^2}$. Hence, we can reformulate the total cost of the supply chain in (r, Q) inventory control under VMI policy according to which Equation (16) is obtained.

$$\begin{aligned} TC^{rQ} = & \frac{F_1+F_2+A_1+A_2+wA_s}{T} + \frac{1}{2}D_1Th_{B1} + \frac{1}{2}D_2Th_{B2} + \frac{1}{2}\left(h_{B1}+h_{B2}\right)z_p\sqrt{\sigma_1^2+\sigma_2^2}\sqrt{\frac{DT}{2P}+\max(b_1,b_2)} \\ & + \frac{1}{2}h_sDT\left(1-\frac{D}{2P}\right) + (1-\alpha)\frac{(h_{B1}+h_{B2})}{2}\sqrt{\sigma_1^2+\sigma_2^2}\sqrt{\frac{DT}{2P}+\max(b_1,b_2)}L^{'}(u) \\ & + \frac{\alpha(\pi_1+\pi_2)+(1-\alpha)(\pi_1^{'}+\pi_2^{'})}{2T}\sqrt{\sigma_1^2+\sigma_2^2}\sqrt{\frac{DT}{2P}+\max(b_1,b_2)}L^{'}(u) \end{aligned} \tag{16}$$

Besides, for finding the optimum value of common cycle time, Equation (16) is derived to acquire Equation (17):

$$\begin{aligned}\frac{\partial TC^{rQ}}{\partial T} &= -\frac{(F_1+F_2+A_1+A_2+wA_s)}{T^2}+\frac{1}{2}D_1h_{B1}+\frac{1}{2}D_2h_{B2}+\frac{1}{2}h_sD\left(1-D/2P\right)\\ &+\left(h_{B1}+h_{B2}\right)\sqrt{\sigma_1^2+\sigma_2^2}\,\frac{D}{8P\sqrt{DT/2P+\max\left(b_1,b_2\right)}}\left(z_p+(1-\alpha)L^{'}(u)\right)\\ &+\left(\alpha(\pi_1+\pi_2)+\left(1-\alpha\right)(\pi_1^{'}+\pi_2^{'})\right)\sqrt{\sigma_1^2+\sigma_2^2}\left[\frac{D}{8PT\sqrt{P+\max\left(b_1,b_2\right)}}-\frac{\sqrt{DT/2P+\max\left(b_1,b_2\right)}}{2T^2}\right]L^{'}(u)\end{aligned} \tag{17}$$

Finally, the optimum value of period length can be calculated using Equation (18).

$$T_{rQ}^{*}=\sqrt{\frac{2(F_1+F_2+A_1+A_2+wA_s)+[\alpha(\pi_1+\pi_2)+\left(1-\alpha\right)(\pi_1^{'}+\pi_2^{'})]\sqrt{\sigma_1^2+\sigma_2^2}\sqrt{(DT/2P)+\max\left(b_1,b_2\right)}L^{'}(u)}{D_1h_{B1}+D_2h_{B2}+h_sD\left(1-D/2P\right)+\left(h_{B1}+h_{B2}\right)z_p\sqrt{\sigma_1^2+\sigma_2^2}\,\frac{D}{4P\sqrt{(DT/2P)+\max\left(b_1,b_2\right)}}+\left[(1-\alpha)(h_{B1}+h_{B2})+\frac{\alpha(\pi_1+\pi_2)+\left(1-\alpha\right)(\pi_1^{'}+\pi_2^{'})}{T}\right]L^{'}(u)\sqrt{\sigma_1^2+\sigma_2^2}\,\frac{D}{4P\sqrt{(DT/2P)+\max\left(b_1,b_2\right)}}}} \tag{18}$$

Here, the subsequent algorithm is employed for attaining the best value of *T* and quantity orders for each buyer:

Step1. Determine the initial value of *T* using $T_1=\sqrt{\frac{2\left(F_1+F_2+A_1+A_2+wA_s\right)}{h_{B1}D_1+h_{B2}D_2+h_sD\left(1-D/2P\right)}}$

Step2. Calculate $\bar{F}\left(r\right)$ and find the *r* using Equation (15).

Step3. Calculate the $L^{'}(u)$ as the right-hand unit of normal linear-loss integral.

Step4. Calculate T_2 by using Equation (18) with replacing of T_1 in the equation.

Step5. If | T_2-T_1|=0,

Denote T_2 by $T_{r,Q}^{*}$

Calculate TC^{rQ} using Equation (16).

Else

Set $T_2 \rightarrow T_1$

Go to Step 2.

Step6. Calculate $Q_1=D_1 T^*_{rQ}$ and $Q_2=D_2 T^*_{rQ}$.

Step7. Stop.

The algorithm concept is similar to the algorithms presented in section 3.1. It starts with an initial period length (T_1) and continues with calculating essential parameters like r, as reorder point, and $L^{'}(u)$. It is noteworthy to mention that the parameter u can be calculated with the equation $u=\dfrac{x-r}{\sigma\sqrt{DT\ /2P+\max\left(b_1,b_2\right)}}$ presented in Appendix B. Also, the algorithm continues until the common cycle time coverage. After obtaining the optimal common cycle time, the total costs of chain and order quantities for both buyers will be calculated in steps 5 and 6.

Figure (1) indicates the flowchart of deriving decision variables in both of (R, T) and (r, Q) replenishment systems.

5. Numerical analysis

This section presents a numerical example borrowed from the literature solved with both of the proposed (R, T) and (r, Q) replenishment reviews under VMI policy when partial back-ordering is permitted. The presented algorithms are employed to solve a model containing one vendor but two buyers. This section aims to represent the applicability of the proposed models and algorithms. Afterward, the sensitivity analysis is done for crucial parameters to represent the significant differences between the proposed VMI system under (R, T) policy and (r, Q). Sensitivity analysis is conducted by varying the parameter values.

Numerical data provided by Rad et al. (2014) has been presented in Table (1). The vendor's weighting factor for ordering cost is assumed 1.7 ($w=1.7$), as described in section 2.1. The percentage of lost sales is supposed to be equal to 0.2 ($1\text{-}\alpha=0.2$). The service level is considered equal to 0.974, so z_p=1.95. The results of the proposed VMI policy with partial back-ordering are shown in Table (2).

Table1. Data for numerical example provided by Rad et al. (2014) which $\boldsymbol{\pi}'$ is added to it.

Parameter	1st buyer	2nd buyer	Parameter	1st buyer	2nd buyer
D	260,000	180,000	h_B	76	70
σ	18,000	12,000	h_s	65	65
P	340,000	340,000	π	200	180
F	200	220	π'	220	210
A_B	180	160	b	0.05	0.04
A_s	320	320			

The proposed algorithms in Section 3.1 and Section 3.2 are applied to obtain the best decision variables for the mentioned example. Algorithms start with the initial common cycle time (Step (1) for both of (*R, T*) and (*r, Q*) algorithms). The solution procedures continue in step (2) to find *F(R)* and *F(r)* as complementary cumulative distributions.

Table2. Results of implementing the proposed VMI system on the numerical example.

Variables	*(R, T)*	*(r, Q)*
TC	1,017,400	986,160
T^*	0.0167	0.0185
Q_1^*	4329	4806
Q_2^*	2997	3327
Backordered	39.0077	34.8960
Lost-Sales	9.7519	8.7240
R/r	27802	24345

Then, $L'(u)$ for the (*R, T*) and (*r, Q*) systems is calculated in step (3). Subsequently, algorithms are continued with step (4) to update common cycle time until algorithms meet the stop conditions declared in step (5). After finding the optimum value for the common cycle times, order quantities for both of the replenishment systems are calculated with step (6). Solution procedures can be tracked with the flowcharts provided in Figure (1). According to Table (2), the aggregate cost of the SC under (*r*, *Q*) model is lower than the (*R, T*). In other words, if all parameters remain unchanged, (*r, Q*) model is considered more conducive for minimizing the total cost of the supply chain in comparison with the (*R*, *T*) model. Furthermore, one can figure out from the above table that (*r*, *Q*) will be a better alternative for industries where responsiveness is a crucial criteria like healthcare industry or medicine supply.

6. Sensitivity analysis

This section provides the impact of changes in main models' parameters, including demand, lost-sales rate, buyer's holding costs, vendor holding cost, cost of back-ordering, cost of lost-sales, and ordering costs on decision variables. Those results are summarized in Tables (3) and (4), which represent the sensitivities of decision variables respect to the parameter changes in (*R*, *T*) and (*r*, *Q*) inventory replenishment systems. Also, changes in the total cost of the chain are shown in Figures (2) to Figure (7), and significant differences between the two replenishment systems are discussed. The purpose of the conducted sensitivity analysis is to compare performances of (*R*, *T*) and (*r*, *Q*) replenishment review systems under VMI strategy.

Figure (2) denotes the percentage of changes in total costs of the chain when buyers' demands are changed. It can be concluded from Figs. (2.a) and (2.b) that the total cost of SC for the mentioned VMI models will be increased with increasing the demand of buyers in both (*r*, *Q*) and (*R*, *T*) systems and vice versa. Also, the (*r*, *Q*) replenishment system shows more sensitivity with changing demands. Thus, implementing (*R*, *T*) replenishment review system under the VMI policy is preferred where demand is more volatile like foods and vegetables or seasonal services such as airline tickets and Recreational Industries.

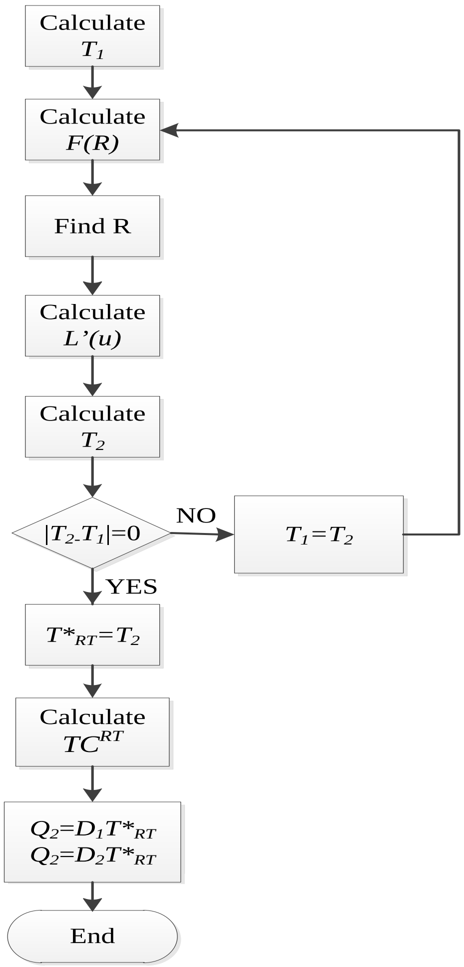

Fig 1.a. Flowchart for deriving (R,T) VMI decision variables

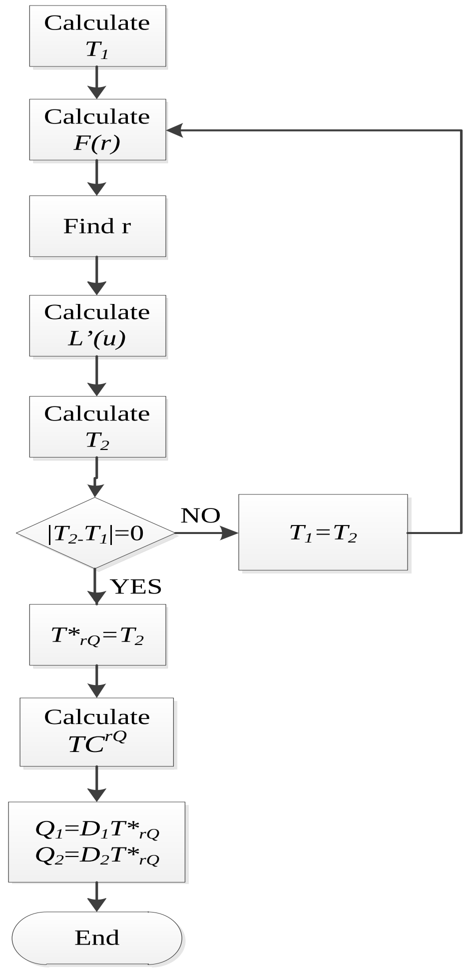

Fig 1.b. Flowchart for deriving (r,Q) VMI decision variables

Figure1. Flowcharts for deriving decision variables in the presented VMI systems.

Table 3. Sensitivity analysis for *(R,T)*.

	Changes	R	T	Q_1	Q_2	Backordered	Lost-Sales	TC(E+06)	%R	%T	%Q_1	%Q_2	%Backordered	%Lost-sales	%TC VMI
	50%	27238	0.0152	5933	2738	38.42	9.60	1.1	-2.0	-9.0	37.0	-8.6	-1.5	-1.5	9.5
	25%	27497	0.0159	5161	2858	38.69	9.67	1.1	-1.1	-4.8	19.2	-4.6	-0.8	-0.8	4.9
D_1	0%	27802	0.0167	4329	2997	39.01	9.75	1.0	-	-	-	-	-	-	-
	-25%	28164	0.0176	3424	3161	39.38	9.84	0.96	1.3	5.4	-20.9	5.5	0.9	0.9	-5.2
	-50%	28606	0.0187	2425	3358	39.81	9.95	0.91	2.9	12.0	-44.0	12.0	2.1	2.1	-10.7
	50%	27422	0.0157	4078	4235	38.61	9.65	1.1	-1.4	-6.0	-5.8	41.3	-1.0	-1.0	6.2
	25%	27602	0.0161	4198	3633	38.80	9.70	1.0	-0.7	-3.6	-3.0	21.2	-0.5	-0.5	3.1
D_2	0%	27802	0.0167	4329	2997	39.01	9.75	1.0	-	-	-	-	-	-	-
	-25%	28025	0.0172	4475	2324	39.24	9.81	0.98	0.8	3.0	3.4	-22.5	0.6	0.6	-3.2
	-50%	28276	0.0178	4639	1606	39.49	9.87	0.95	1.7	6.6	7.1	-46.4	1.2	1.2	-6.6
	25%	27801	0.0166	4303	2979	48.70	0.00	1.0	0.0	-0.6	-0.6	-0.6	24.9	-100.0	-1.4
	0%	27802	0.0167	4329	2997	39.01	9.75	1.0	-	-	-	-	-	-	-
α	-25%	27802	0.0167	4354	3015	29.29	19.53	1.0	0.0	0.0	0.6	0.6	-24.9	100.2	1.4
	-50%	27802	0.0168	4379	3031	19.55	29.32	1.0	0.0	0.6	1.1	1.1	-49.9	200.7	2.8
	50%	27234	0.0152	4.0	2737	38.39	9.60	1.1	-2.0	-9.0	-8.7	-8.7	-1.6	-1.6	9.8
	25%	27496	0.0159	4.1	2858	38.68	9.67	1.1	-1.1	-4.8	-4.7	-4.7	-0.8	-0.8	5.0
h_{B1}	0%	27802	0.0167	4.3	2997	39.01	9.75	1.0	-	-	-	-	-	-	-
	-25%	28166	0.0176	4.6	3162	39.39	9.85	0.96	1.3	5.4	5.5	5.5	1.0	1.0	-5.3
	-50%	28610	0.0187	4.9	3360	39.85	9.96	0.91	2.9	12.0	12.1	12.1	2.2	2.2	-11.0
	50%	27419	0.0157	3954	2822	38.58	9.65	1.1	-1.4	-6.0	-5.8	-5.8	-1.1	-1.1	6.5
	25%	27600	0.0161	4128	2906	38.79	9.70	1.1	-0.7	-3.6	-3.1	-3.1	-0.6	-0.6	3.3
h_{B2}	0%	27802	0.0167	4329	2997	39.01	9.75	1.0	-	-	-	-	-	-	-
	-25%	28027	0.0172	4567	3099	39.25	9.81	0.98	0.8	3.0	3.4	3.4	0.6	0.6	-3.4
	-50%	28280	0.0178	4853	3213	39.52	9.88	0.95	1.7	6.6	7.2	7.2	1.3	1.3	-6.9

Table 3. Sensitivity analysis for *(R,T) (Continue)*.

	Changes	R	T	Q_1	Q_2	Backordered	Lost-Sales	TC(E+06)	%R	%T	$\%Q_1$	$\%Q_2$	%Back ordered	%Lost-sales	%TC VMI
	50%	27491	0.0159	2855	2855	38.69	9.67	1.1	-1.1	-4.8	-4.7	-4.7	-0.8	-0.8	5.0
	25%	27640	0.0162	2924	2924	38.84	9.71	1.0	-0.6	-3.0	-2.5	-2.5	-0.4	-0.4	2.5
h_s	0%	27802	0.0167	2997	2997	39.01	9.75	1.0	-	-	-	-	-	-	-
	-25%	27978	0.0171	3077	3077	39.19	9.80	0.99	0.6	2.4	2.7	2.7	0.5	0.5	-2.6
	-50%	28173	0.0176	3165	3165	39.38	9.85	0.96	1.3	5.4	5.6	5.6	1.0	1.0	-5.3
	50%	27804	0.0174	3127	3127	39.33	9.83	1.1	0.0	4.2	4.3	4.3	0.8	0.8	10.4
	25%	27803	0.017	3065	3065	39.18	9.79	1.1	0.0	1.8	2.3	2.3	0.4	0.4	5.3
π_1	0%	27802	0.0167	2997	2997	39.01	9.75	1.0	-	-	-	-	-	-	-
	-25%	27800	0.0162	2921	2921	38.82	9.70	0.96	0.0	-3.0	-2.5	-2.5	-0.5	-0.5	-5.4
	-50%	27798	0.0158	2836	2836	38.61	9.65	0.91	0.0	-5.4	-5.4	-5.4	-1.0	-1.0	-10.9
	50%	27804	0.0173	3115	3115	39.30	9.82	1.1	0.0	3.6	3.9	3.9	0.7	0.7	9.4
	25%	27803	0.017	3059	3059	39.16	9.79	1.1	0.0	1.8	2.1	2.1	0.4	0.4	4.7
π_2	0%	27802	0.0167	2997	2997	39.01	9.75	1.0	-	-	-	-	-	-	-
	-25%	27800	0.0163	2929	2929	38.84	9.71	0.97	0.0	-2.4	-2.3	-2.3	-0.4	-0.4	-4.8
	-50%	27799	0.0159	2854	2854	38.65	9.66	0.92	0.0	-4.8	-4.8	-4.8	-0.9	-0.9	-9.8
	50%	27802	0.0169	3036	3036	39.10	9.78	1.0	0.0	1.2	1.3	1.3	0.2	0.2	2.9
	25%	27802	0.0168	3017	3017	39.06	9.76	1.0	0.0	0.6	0.6	0.6	0.1	0.1	1.5
π'_1	0%	27802	0.0167	2997	2997	39.01	9.75	1.0	-	-	-	-	-	-	-
	-25%	27801	0.0165	2977	2977	38.96	9.74	1.0	0.0	-1.2	-0.7	-0.7	-0.1	-0.1	-1.5
	-50%	27801	0.0164	2956	2956	38.91	9.73	0.99	0.0	-1.8	-1.4	-1.4	-0.3	-0.3	-2.9
	50%	27802	0.0169	3034	3034	39.10	9.77	1.0	0.0	1.2	1.2	1.2	0.2	0.2	2.8
	25%	27802	0.0168	3016	3016	39.05	9.76	1.0	0.0	0.6	0.6	0.6	0.1	0.1	1.4
π'_2	0%	27802	0.0167	2997	2997	39.01	9.75	1.0	-	-	-	-	-	-	-
	-25%	27801	0.0165	2978	2978	38.96	9.74	1.0	0.0	-1.2	-0.6	-0.6	-0.1	-0.1	-1.4
	-50%	27801	0.0164	2958	2958	38.91	9.73	0.99	0.0	-1.8	-1.3	-1.3	-0.2	-0.2	-2.8

Table 3. Sensitivity analysis for *(R,T) (Continue)*.

	Changes	**R**	**T**	**Q_1**	**Q_2**	**Backordered**	**Lost-Sales**	**TC (E+06)**	**%R**	**%T**	**%Q_1**	**%Q_2**	**%Backordered**	**%Lost-sales**	**%TC VMI**
	50%	27994	0.0169	4387	3037	39.10	9.77	1.0	0.7	1.2	1.3	1.3	0.2	0.2	0.3
	25%	27899	0.0168	4359	3018	39.05	9.76	1.0	0.3	0.6	0.7	0.7	0.1	0.1	0.1
A_1	0%	27802	0.0167	4329	2997	39.01	9.75	1.0	-	-	-	-	-	-	-
	-25%	27703	0.0165	4299	2976	38.96	9.74	1.0	-0.4	-1.2	-0.7	-0.7	-0.1	-0.1	-0.1
	-50%	27602	0.0164	4268	2955	38.91	9.73	1.0	-0.7	-1.8	-1.4	-1.4	-0.2	-0.2	-0.3
	50%	27973	0.0168	4381	3033	39.09	9.77	1.0	0.6	0.6	1.2	1.2	0.2	0.2	0.2
	25%	27888	0.0168	4355	3015	39.05	9.76	1.0	0.3	0.6	0.6	0.6	0.1	0.1	0.1
A_2	0%	27802	0.0167	4329	2997	39.01	9.75	1.0	-	-	-	-	-	-	-
	-25%	27714	0.0165	4303	2979	38.97	9.74	1.0	-0.3	-1.2	-0.6	-0.6	-0.1	-0.1	-0.1
	-50%	27625	0.0164	4275	2960	38.93	9.73	1.0	-0.6	-1.8	-1.2	-1.2	-0.2	-0.2	-0.2
	50%	28366	0.0173	4498	3114	39.26	9.82	1.0	2.0	3.6	3.9	3.9	0.7	0.7	0.9
	25%	28090	0.017	4416	3057	39.14	9.79	1.0	1.0	1.8	2.0	2.0	0.3	0.3	0.4
A_s	0%	27802	0.0167	4329	2997	39.01	9.75	1.0	-	-	-	-	-	-	-
	-25%	27497	0.0163	4236	2933	38.87	9.72	1.0	-1.1	-2.4	-2.2	-2.2	-0.4	-0.4	-0.4
	-50%	27175	0.0159	4135	2863	38.71	9.68	1.0	-2.3	-4.8	-4.5	-4.5	-0.8	-0.8	-0.7

Table 4. Sensitivity analysis for *(r,Q)*.

	Changes	r	T	Q_1	Q_2	Backordered	Lost-Sales	TC(E+06)	%r	%T	%Q_1	%Q_2	%Backordered	%Lost-sales	%TCVMI
	50%	24124	0.0168	6552	3024	34.60	8.65	1.1	-0.9	-9.2	36.3	-9.1	-0.9	-0.9	9.8
	25%	24226	0.0176	5714	3165	34.74	8.68	1.0	-0.5	-4.9	18.9	-4.9	-0.5	-0.5	5.0
D_1	0%	24345	0.0185	4807	3328	34.90	8.72	0.99	-	-	-	-	-	-	-
	-25%	24487	0.0196	3814	3521	35.08	8.77	0.93	0.6	5.9	-20.7	5.8	0.5	0.5	-5.3
	-50%	24660	0.0209	2711	3753	35.31	8.83	0.88	1.3	13.0	-43.6	12.8	1.2	1.2	-11.0
	50%	24197	0.0174	4512	4686	34.70	8.67	1.0	-0.6	-5.9	-6.1	40.8	-0.6	-0.6	6.4
	25%	24267	0.0179	4652	4026	34.79	8.70	1.0	-0.3	-3.2	-3.2	21.0	-0.3	-0.3	3.2
D_2	0%	24345	0.0185	4807	3328	34.90	8.72	0.99	-	-	-	-	-	-	-
	-25%	24432	0.0191	4979	2585	35.01	8.75	0.95	0.4	3.2	3.6	-22.3	0.3	0.3	-3.4
	-50%	24531	0.0199	5171	1790	35.14	8.79	0.92	0.8	7.6	7.6	-46.2	0.7	0.7	-6.9
	25%	24345	0.0183	4767	3300	43.58	0.00	0.99	0.0	-1.1	-0.8	-0.8	24.9	-100.0	0.1
α	0%	24345	0.0185	4807	3328	34.90	8.72	0.99	-	-	-	-	-	-	-
	-25%	24345	0.0186	4846	3355	26.19	17.46	0.99	0.0	0.5	0.8	0.8	-24.9	100.2	-0.1
	-50%	24346	0.0188	4884	3381	17.48	26.22	0.98	0.0	1.6	1.6	1.6	-49.9	200.5	-0.2
	50%	24121	0.0168	4366	3023	34.57	8.64	1.1	-0.9	-9.2	-9.1	-9.1	-0.9	-0.9	10.1
	25%	24224	0.0176	4570	3164	34.72	8.68	1.0	-0.5	-4.9	-4.9	-4.9	-0.5	-0.5	5.2
h_{B1}	0%	24345	0.0185	4806	3327	34.90	8.72	0.99	-	-	-	-	-	-	-
	-25%	24489	0.0196	5087	3522	35.10	8.77	0.93	0.6	5.9	5.8	5.8	0.6	0.6	-5.5
	-50%	24664	0.0209	5424	3755	35.34	8.84	0.87	1.3	13.0	12.9	12.9	1.3	1.3	-11.4
	50%	24194	0.0173	4511	3123	34.67	8.67	1.1	-0.6	-6.5	-6.1	-6.1	-0.6	-0.6	6.7
	25%	24265	0.0179	4651	3220	34.78	8.69	1.0	-0.3	-3.2	-3.2	-3.2	-0.3	-0.3	3.4
h_{B2}	0%	24345	0.0185	4806	3327	34.90	8.72	0.99	-	-	-	-	-	-	-
	-25%	24434	0.0192	4980	3448	35.03	8.76	0.95	0.4	3.8	3.6	3.6	0.4	0.4	-3.5
	-50%	24535	0.0199	5174	3582	35.17	8.79	0.92	0.8	7.6	7.6	7.7	0.8	0.8	-7.2

Table 4. Sensitivity analysis for *(r,Q) (Continue)*.

	Changes	r	T	Q_1	Q_2	Backordered	Lost-Sales	TC(E+06)	%r	%T	$\%Q_1$	$\%Q_2$	%Back ordered	%Lost-sales	%TCVMI
	50%	24224	0.0176	4566	3161	34.73	8.68	1.0	-0.5	-4.9	-5.0	-5.0	-0.5	-0.5	5.1
	25%	24282	0.018	4682	3241	34.81	8.70	1.0	-0.3	-2.7	-2.6	-2.6	-0.2	-0.2	2.6
h_s	0%	24345	0.0185	4806	3327	34.90	8.72	0.99	-	-	-	-	-	-	-
	-25%	24414	0.019	4943	3422	34.99	8.75	0.96	0.3	2.7	2.8	2.8	0.3	0.3	-2.7
	-50%	24490	0.0196	5092	3525	35.09	8.77	0.93	0.6	5.9	5.9	5.9	0.6	0.6	-5.5
	50%	24348	0.0196	5101	3531	35.12	8.78	1.1	0.0	5.9	6.1	6.1	0.6	0.6	10.0
	25%	24346	0.0191	4959	3433	35.01	8.75	1.0	0.0	3.2	3.2	3.2	0.3	0.3	5.1
π_1	0%	24345	0.0185	4806	3327	34.90	8.72	9.9	-	-	-	-	-	-	-
	-25%	24344	0.0179	4642	3214	34.77	8.69	9.3	0.0	-3.2	-3.4	-3.4	-0.4	-0.4	-5.2
	-50%	24342	0.0172	4463	3090	34.63	8.66	8.8	0.0	-7.0	-7.2	-7.1	-0.8	-0.8	-10.6
	50%	24347	0.0195	5073	3512	35.10	8.77	1.1	0.0	5.4	5.5	5.5	0.6	0.6	9.0
	25%	24346	0.019	4944	3423	35.00	8.75	1.0	0.0	2.7	2.9	2.9	0.3	0.3	4.6
π_2	0%	24345	0.0185	4806	3327	34.90	8.72	0.99	-	-	-	-	-	-	-
	-25%	24344	0.0179	4659	3226	34.78	8.70	0.94	0.0	-3.2	-3.1	-3.1	-0.3	-0.3	-4.7
	-50%	24342	0.0173	4500	3116	34.66	8.66	0.89	0.0	-6.5	-6.4	-6.4	-0.7	-0.7	-9.5
	50%	24346	0.0188	4892	3387	34.96	8.74	1.0	0.0	1.6	1.8	1.8	0.2	0.2	2.4
	25%	24345	0.0187	4850	3358	34.93	8.73	1.0	0.0	1.1	0.9	0.9	0.1	0.1	1.2
π'_1	0%	24345	0.0185	4806	3327	34.90	8.72	0.99	-	-	-	-	-	-	-
	-25%	24345	0.0183	4763	3297	34.86	8.72	0.97	0.0	-1.1	-0.9	-0.9	-0.1	-0.1	-1.2
	-50%	24344	0.0181	4718	3266	34.83	8.71	0.96	0.0	-2.2	-1.8	-1.8	-0.2	-0.2	-2.5
	50%	24346	0.0188	4888	3384	34.96	8.74	1.0	0.0	1.6	1.7	1.7	0.2	0.2	2.3
	25%	24345	0.0186	4848	3356	34.93	8.73	1.0	0.0	0.5	0.9	0.9	0.1	0.1	1.2
π'_2	0%	24345	0.0185	4806	3327	34.90	8.72	0.99	-	-	-	-	-	-	-
	-25%	24345	0.0183	4765	3299	34.86	8.72	0.97	0.0	-1.1	-0.9	-0.9	-0.1	-0.1	-1.2
	-50%	24344	0.0182	4722	3269	34.83	8.71	0.96	0.0	-1.6	-1.8	-1.8	-0.2	-0.2	-2.4

Table 4. Sensitivity analysis for *(r,Q)(continue)*.

	Changes	r	T	Q_1	Q_2	Backordered	Lost-Sales	TC(E+06)	%r	%T	$\%Q_1$	$\%Q_2$	%Backordered	%Lost-sales	%TCVMI
	50%	24421	0.0187	4852	3359	34.92	8.73	0.99	0.3	1.1	0.9	0.9	0.1	0.1	0.4
	25%	24383	0.0186	4829	3343	34.91	8.73	0.99	0.2	0.5	0.5	0.5	0.0	0.0	0.2
A_1	0%	24345	0.0185	4806	3327	34.90	8.72	0.99	-	-	-	-	-	-	-
	-25%	24306	0.0184	4784	3312	34.88	8.72	0.98	-0.2	-0.5	-0.5	-0.5	0.0	0.0	-0.2
	-50%	24267	0.0183	4761	3296	34.87	8.72	0.98	-0.3	-1.1	-1.0	-1.0	-0.1	-0.1	-0.4
	50%	24412	0.0186	4847	3355	34.92	8.73	0.99	0.3	0.5	0.8	0.8	0.1	0.1	0.4
	25%	24379	0.0186	4827	3342	34.91	8.73	0.99	0.1	0.5	0.4	0.4	0.0	0.0	0.2
A_2	0%	24345	0.0185	4806	3327	34.90	8.72	0.99	-	-	-	-	-	-	-
	-25%	24311	0.0184	4787	3314	34.88	8.72	0.98	-0.1	-0.5	-0.4	-0.4	0.0	0.0	-0.2
	-50%	24276	0.0183	4766	3299	34.87	8.72	0.98	-0.3	-1.1	-0.9	-0.9	-0.1	-0.1	-0.3
	50%	24566	0.019	4937	3418	34.98	8.74	1.0	0.9	2.7	2.7	2.7	0.2	0.2	1.2
	25%	24458	0.0187	4874	3374	34.94	8.73	0.99	0.5	1.1	1.4	1.4	0.1	0.1	0.6
A_s	0%	24345	0.0185	4806	3327	34.90	8.72	0.99	-	-	-	-	-	-	-
	-25%	24226	0.0182	4736	3279	34.85	8.71	0.98	-0.5	-1.6	-1.5	-1.5	-0.1	-0.1	-0.6
	-50%	24100	0.0179	4660	3226	34.80	8.70	0.97	-1.0	-3.2	-3.1	-3.1	-0.3	-0.3	-1.2

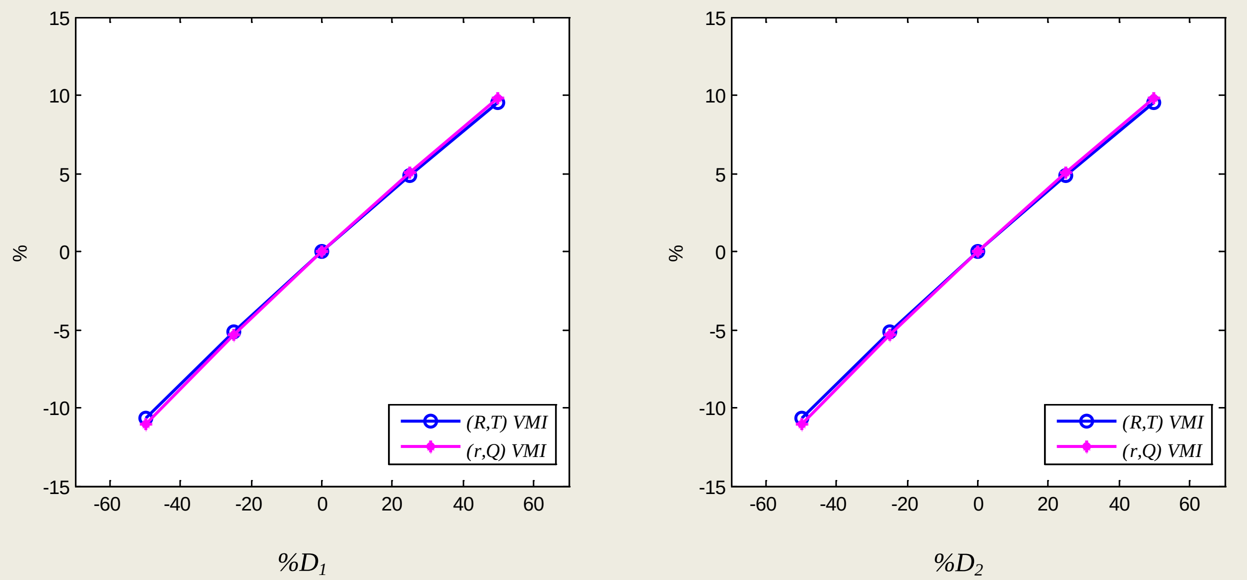

Figure 2. Changes of the total SC cost based on a variation in first and second buyer's demand.

On the other hand, (*r, Q*) will remain more responsive according to the obtained results, in which the level of lost-sales and backorders are less than that of (*R, T*). Thus, the vendor can enhance the responsiveness of the chain by implementing the VMI system under (*r, Q*) policy. As can be seen from the data provided in Table (3) and Table (4), increasing in buyer's demand results in reducing the common cycle time in both replenishment review systems. That is because the vendor tends to meet demand more reliably when demands are increased. Therefore, less common cycle time is required to produce lot-sizes. In other words, the vendor should produce goods and ship them to buyers sooner with a common delivery time.

Nevertheless, increasing demand may reduce the maximum inventory level in (*R, T*), and the point of reordering under (*r, Q*) replenishment review systems. Furthermore, it is evident from the obtained results that how changing demand for one of the buyers can affect the optimal decision for the second buyer and the whole chain. Noteworthy to mention that optimal decisions may not be changed in other traditional management systems (RMI) during changes in buyer's demand because the vendor and buyers act separately in order to reduce their own inventory cost without sharing information.

Figure (3) represents changes in the total cost of the chain with changing buyers' holding costs. As can be seen in Figs (3.a) and (3.b), increasing in holding cost of the buyers, results in increasing the total costs of the chain in (*r, Q*) and (*R, T*) systems, and vice versa. It is noteworthy that the (*r, Q*) system is more sensitive setting with respect to the changes in holding costs. So, implementing the proposed (*r, Q*) replenishment system under VMI is more lucrative for the products with lower holding costs (for example items like books and clothes).

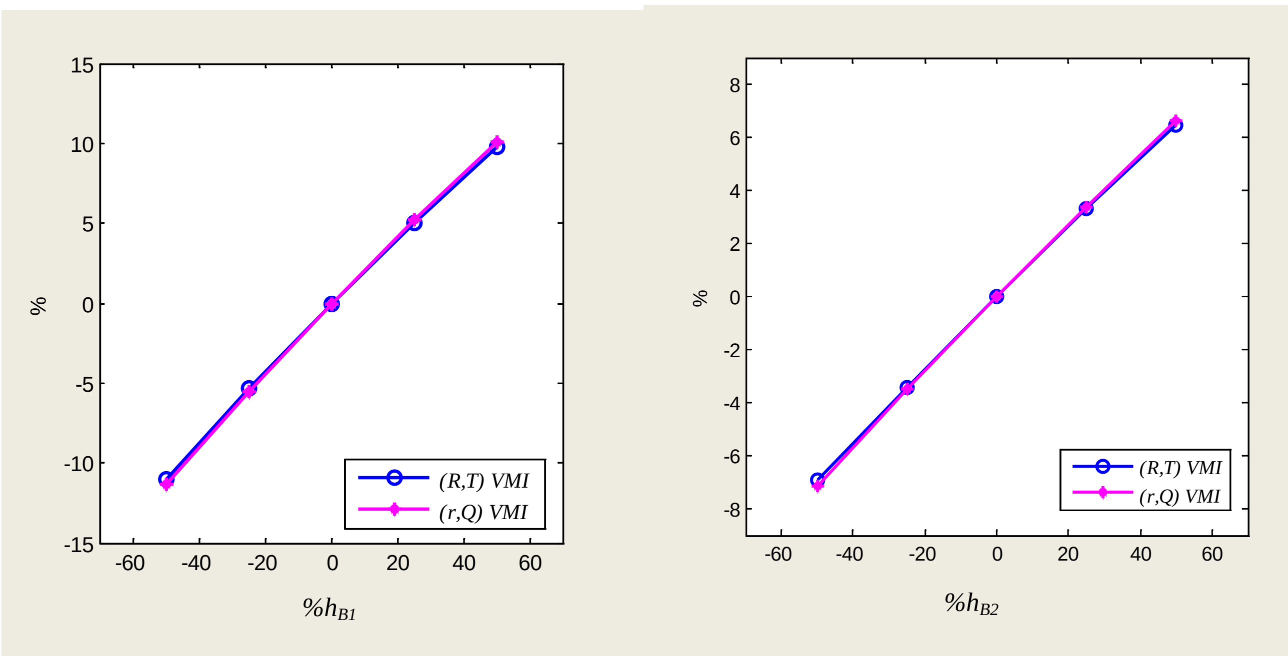

Figure 3. Changes in the total cost of the SC based on a variety of changes in the first and second buyer's holding cost.

Otherwise, if buyers may encounter higher holding costs (for instance deteriorating items like foods and vegetables), implementing the (*R*, *T*) could be more beneficial for the chain. Furthermore, increasing buyers' holding cost results in decreasing expected orders quantity in (*R*, *T*) and (*r*, *Q*), which can be concluded from Table (3) and Table (4). Therefore, the vendor will produce fewer quantities with shorter delivery times. That is why the chain would gain more profit with fewer inventory levels instead of producing more and paying more for holding costs. Also, it can be concluded that the common cycle will be reduced in both of the replenishment systems with increasing buyer's holding cost.

Fig. (4.a) represents an increase in the total cost of chain with increasing in holding cost of the vendor. Also, decision variables under (*r*, *Q*) system are more sensitive than (*R*, *T*) system decision variables concerning the changes in the vendor's holding cost. Besides, the maximum inventory level of (*R*, *T*) will decrease with the reduction in holding cost of vendor, meaning that it would be more profitable if the vendor tries to reduce the holding cost. This action results in reducing the aggregate cost of SC. On the other hand, reduction in holding cost of the vendor will lead to reducing the reorder point for the (*r*, *Q*) replenishment systems, which means it is more likely to hold fewer inventories on the side of buyers. Hence, implementing VMI with (*r*, *Q*) replenishment system could enhance the cost efficiency of the chain if the vendor tries to decrease its holding cost. Otherwise (*R*, *T*) is more preferred.

Fig (4.b) represents the effect of changing back-ordering rate (α) on the total cost of the chain in (*r*, *Q*) and (*R*, *T*) replenishment reviews with the proposed VMI system. Increasing the rate of back-ordering, results in decreasing the total costs of the chain in (*R*, *T*) replenishment review system. Nonetheless, increasing the rate of back-ordering, leads to increasing the total cost of chain in (*r*, *Q*) replenishment review system. In accordance, implementing the proposed (*R*, *T*) system can enhance the cost efficiency of the chain by increasing the backorder rate and consequently decreasing the lost-sales rate.

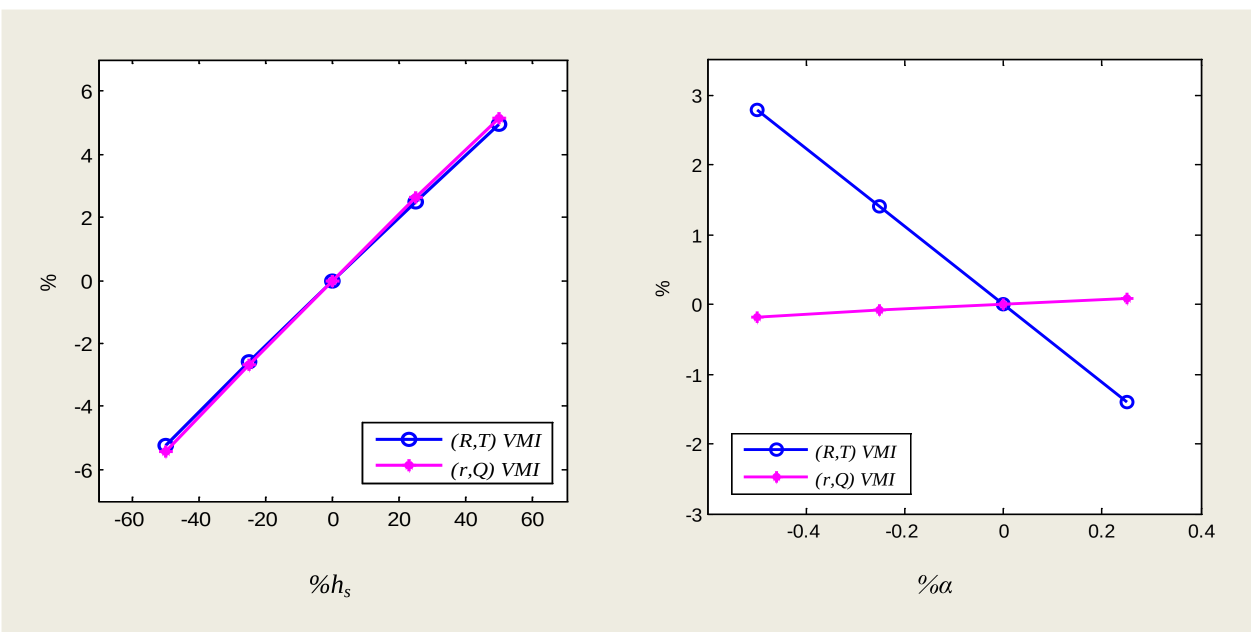

Figure 4. SC total cost changes based on changes in holding cost of vendor and back-ordering rate.

If one wants to implement the proposed VMI system under (*R*, *T*), it will be more profitable by facing the demands with backorders. On the contrary, implementing the proposed VMI system under (*r*, *Q*) replenishment review with more amount of lost-sales can be profitable. Accordingly, if an inventory management system is more likely to be encountered with lost-sales, the proposed VMI with (*r*, *Q*) replenishment review system will be preferred.

Figure (5) demonstrates how the two replenishment review system behave when buyers' back-ordering costs vary. Increasing buyer's back-ordering cost leads to increasing the total costs of the chain in (*R*, *T*) and (*r*, *Q*) replenishment review systems. Note that (*R*, *T*) is more sensitive than (*r*, *Q*) when the back-ordering costs increase or decrease. Where buyers' back-ordering cost may be higher, it is expected to implement the proposed VMI system under (*r*, *Q*) replenishment review policy. Furthermore, if buyers may reduce their back-ordering costs, implementing (*R*, *T*) will be preferred compared to (*r*, *Q*) because the cost efficiency of the chain will be improved much better. Also, as can be seen from Table (3) and Table (4), reducing buyers' back-ordering cost will lead to a decrease in the common cycle times in both of the replenishment systems, and consequently, the delivery time will be reduced. Thus, buyers will be provided with finished goods, with shorter common cycle time and delivery time.

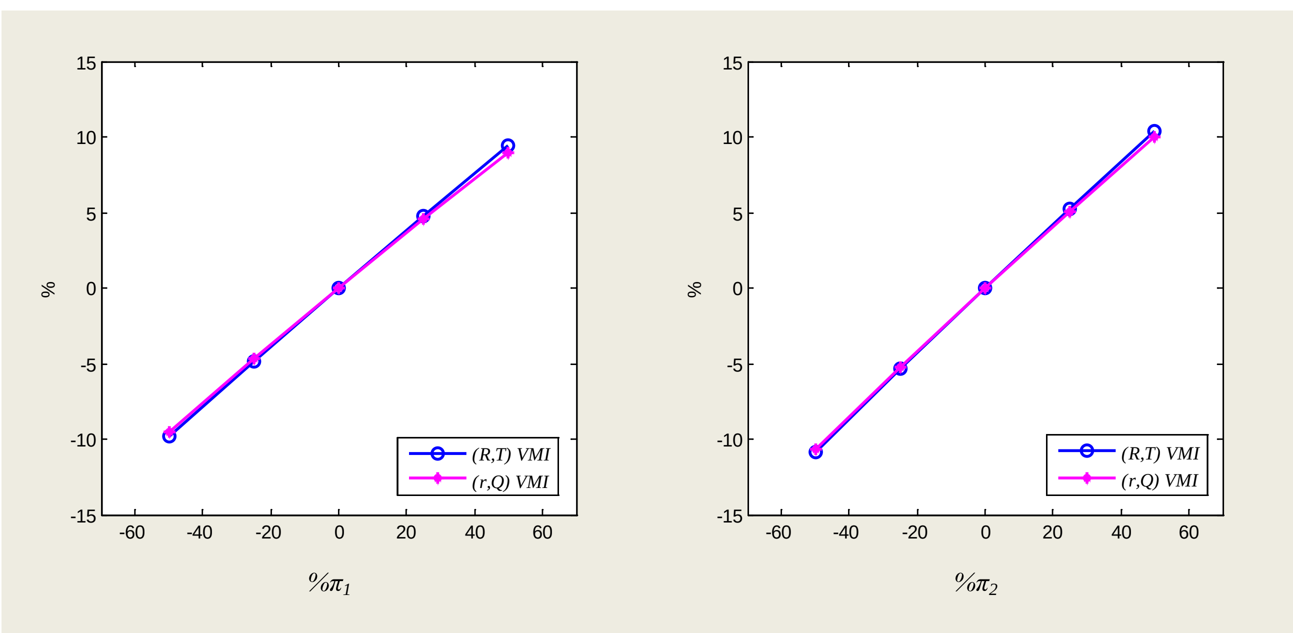

Figure 5. Changes in the total cost of the chain based on changes of first and second buyer’s back-ordering cost.

Fig. (6.a) and Fig. (6.b) are related to changing buyers’ lost-sales costs. Accordingly, the total cost of the SC will be increased when buyers’ lost-sales costs increase. However, it can be concluded that (*R*, *T*) is more sensitive than (*r*, *Q*) to lost-sales costs variations similar to the buyers’ lost-sales costs.

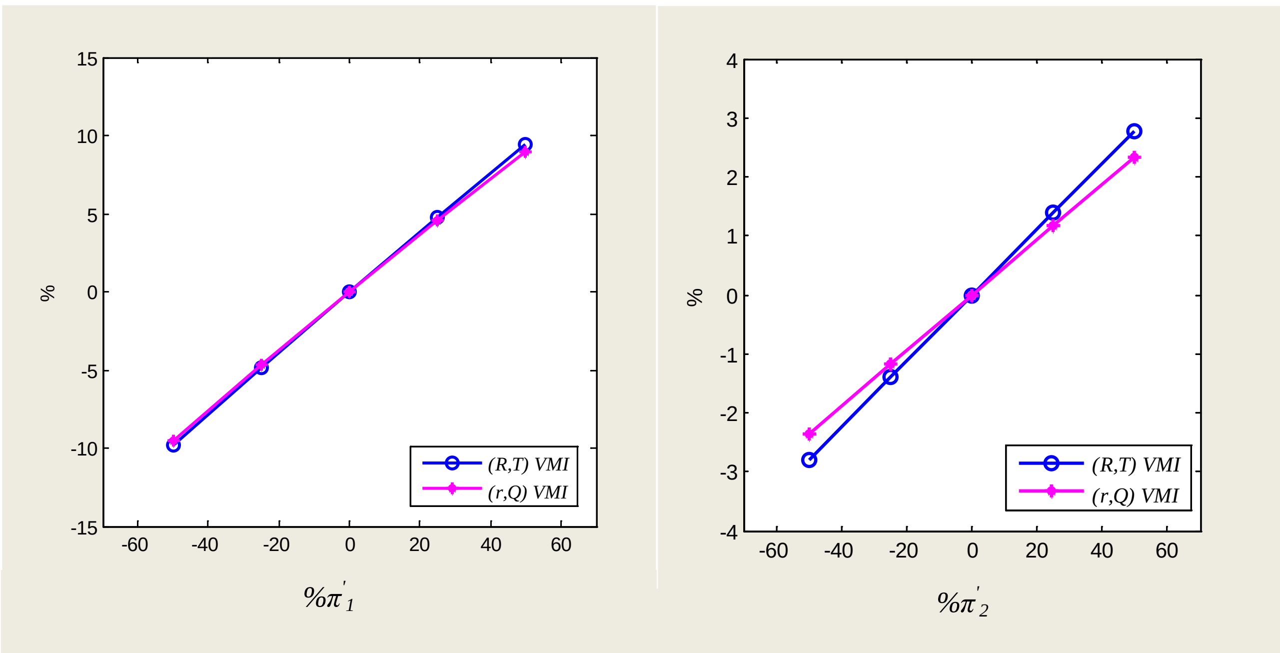

Figure 6. Changes in the total cost of the chain based on changes in buyers’ lost-sales costs.

As a result, when buyers may encounter higher lost-sales costs, it is more profitable to implement the proposed VMI system under (*r, Q*), which imposes fewer costs to the whole chain in comparison with (*R*, *T*) replenishment review system. If buyers are capable of reducing their lost-sales costs, (*R*, *T*) will be preferred in comparison with the (*r*, *Q*). According to the Table (3) and Table (4), reduction in buyers’ lost-sales costs will bring about a reduction in the common cycle time of (*r*, *Q*) and (*R*, *T*) policies. Thus, the vendor can provide buyers with goods faster, resulting from improving responsiveness and cost-efficiency of the chain.

Figure (7) shows the results of variations in ordering cost of buyers and the vendor on the total cost of chain. Increasing ordering costs for buyers, and the vendor leads to an increase in the aggregate cost of SC. As can be seen, (*R*, *T*) is less sensitive than (*r*, *Q*) with ordering cost variations. As can be concluded from Figure (7) in inventory management systems where ordering costs may be higher (i.e., Gas distribution industries), implementing the proposed VMI system under (*R*, *T*) is more preferred than (*r*, *Q*). Conversely, using an electronic data interchange (EDI) system, which helps the cost of the order to be reduced, applying the VMI system under (*r*, *Q*) can be beneficial.

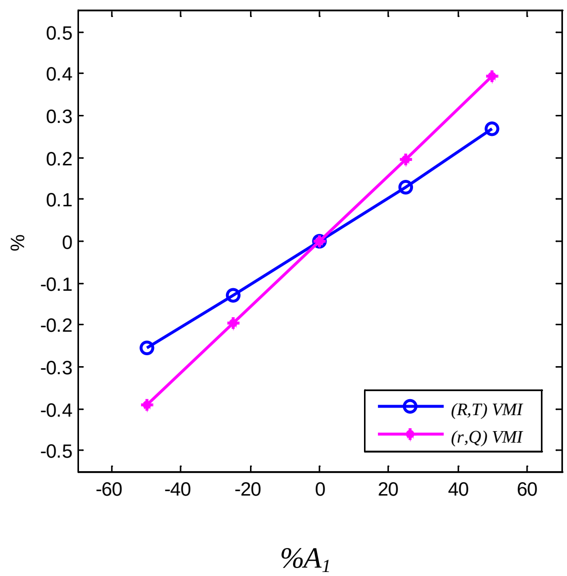

Fig. 7.a. Changes in first buyer's ordering cost

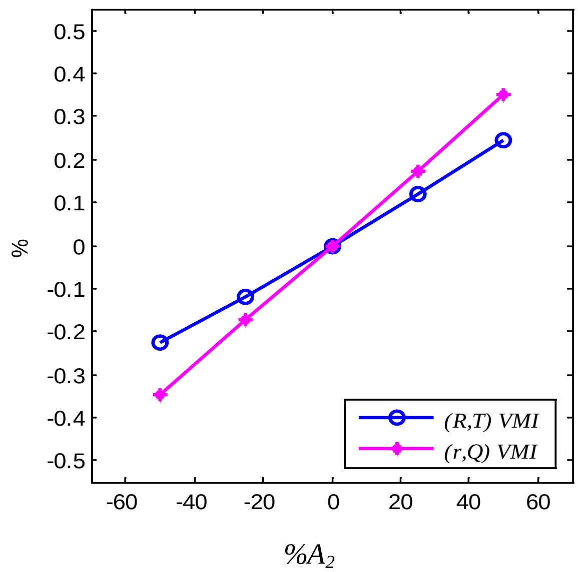

Fig. 7.b. Changes in second buyer's ordering cost

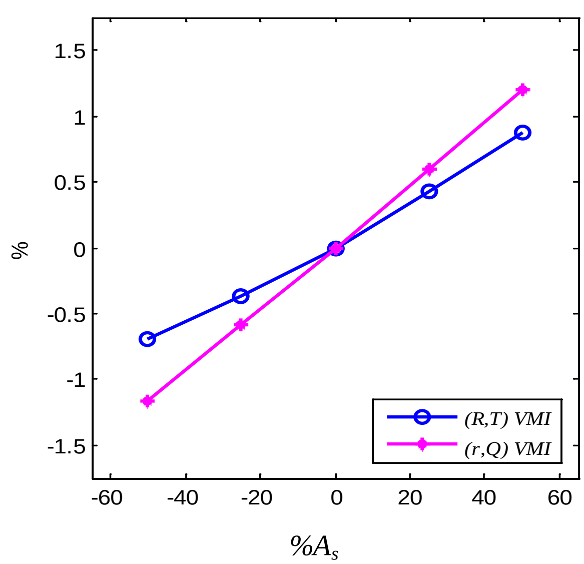

Fig. 7.c. Changes in vendor's cost of ordering

Figure 7. Changes of the aggregate cost of SC based on changes in buyers' and vendor's ordering costs.

Furthermore, as can be seen from Table (3) increasing in buyers and vendor ordering costs may lead to an increase in the maximum level of inventory in (*R*, *T*). That is because orders with higher ordering costs should be placed to decrease the overall cost of ordering and there is a need for holding more inventories. Thus, the common cycle time will be increased and thereby, the number of orders will be decreased. Moreover, Table (4) demonstrates how the common cycle time will be reduced in (*r, Q*) model. In other words, fewer ordering costs may increase the number of orders in both of the (*r, Q*) and (*R, T*) systems. As a result of reducing the common cycle for both of the buyers, the common delivery time will be reduced as well, leading to enhanced responsiveness and cost-efficiency of the whole chain. Therefore, it is useful to exploit methods to reduce the ordering costs such as the weighting factor method (Rad et al., 2014) using in this paper to develop models.

It is noteworthy to mention that, in this paper, the weighting factor of the cost of ordering for the vendor (Rad et al., 2014) is utilized for decreasing the overall cost of ordering in VMI. As a result, reducing ordering costs with such methods can be worthwhile when the (*R*, *T*) and (*r, Q*) replenishment review systems implement under VMI policy simultaneously with partial back-ordering. What is more, it can be concluded from Table (3) and Table (4) that the total cost of the chain under the proposed VMI for (*r, Q*) is lower than (*R, T*). Consequently, (*r, Q*) may lead the chain to be more cost-efficient if the parameters such as demands, holding costs, order costs, backorder costs, lost-sales costs, and back-ordering rates remain constant.

7. Managerial Insights

In this research paper, according to the numerical example provided above and sensitivity analysis, the main findings of the paper have been outlined in this section. The main conclusions of this work are presented concerning eight criteria through which one can find which model ((R, T) or (r, Q)) is preferable for cost reduction. The following results have been summarized in Table (5):

- ***Demand:*** The higher amounts of demand lead to higher cost function both in (R, T) and (r, Q). However, for industries and products with high fluctuations in demand, (R, T) would be a better option.
- ***Buyer's holding cost:*** The more is the holding cost, the more is the value of cost for both (R, T) and (r, Q). However, with commodities with greater holding costs, (R, T) would be cost-efficient.
- ***Vendor's holding cost:*** The greater amounts of holding cost will result in higher amounts of the cost function. Nonetheless, for inventory systems with lower holding costs, (r, Q) is desired while when holding costs is high, (R, T) is a much better alternative.
- ***Back-ordering rate:*** When the rate of back-ordering increases, the total cost in (R, T) will decrease, whereas the total cost in (r, Q) will increase. Hence, for the inventory systems with a high probability of facing lost-sales, (r, Q) will be more lucrative.
- ***Ordering cost:*** The more is the ordering costs, the more is the total cost. It is noteworthy that for inventory systems with high ordering costs, (R, T) is preferred, but for systems with lower ordering costs, (r, Q) is a better choice.
- ***Lost-sales and back-ordering cost:*** When lost-sales or back-ordering costs rise, the total cost of the supply chain will increase drastically. Despite this, for commodities with lower lost-sale or back-ordering costs, (R, T) would be a better decision. On the opposite side, for inventory systems that lost-sale or back-ordering will penalize the system very much, (r, Q) behaves more efficiently regarding the cost value.

- ***Responsiveness:*** As stated in the numerical example, (*r*, *Q*) is better for the supply chain regarding responsiveness since the amount of lost-sales and back-ordering is much lower in (*r*, *Q*) compared to the (*R*, *T*) policy.
- ***Aggregate cost-efficiency:*** It was shown in the numerical example that (*r*, *Q*) model acts more cost-efficiently if all parameters remain unchanged.

Table 5. Decision-making framework for (*R*, *T*) and (*r*, *Q*) under VMI with partial back-ordering.

(r,Q)	*(R,T)*
Low fluctuated demand	High fluctuated demand
Low holding costs	High holding costs
High chance of facing lost-sales	Low chance of facing lost-sales
Low ordering costs	High ordering costs
High lost-sale or back-ordering costs	Low lost-sale or back-ordering costs
Responsiveness is vital	Responsiveness is not that important
To minimize total SC cost	N/A

8. Conclusion

Firms benefit from implementing VMI programs, which helps them to be more competitive and enhance relationships and effectiveness among all parts of a supply chain (Yao and Dresner, 2008). Buyers or retailers feed suppliers with online inventory data, and the supplier makes replenishment decisions for them (Rad et al., 2014). Reduced cost and improved service are some advantages of implementing VMI (Waller et al., 1999).

In this work, a VMI model in a two-echelon supply chain, including one vendor and two buyers are considered to develop periodic replenishment (R, T) and continuous replenishment (r, Q) models with partial back-ordering under VMI policy. In partial back-ordering, lost-sales and back-ordering are allowed as this assumption is more pragmatic. Therefore, the weighting factor of the vendor's cost of the order is utilized in the modeling. Decision variables (replenishment decisions) are derived in both of the presented models, and two algorithms are proposed as solution procedures to determine their optimal values. A numerical example is used to depict the applications of the generated models. Finally, sensitivity analysis is done for the critical parameters, and some significant disparities between two replenishment policies are investigated accordingly.

As concluded from the provided sensitivity analysis, both the (r, Q) and (R, T) replenishment systems have some pros and cons in different inventory systems or for various objectives. For instance, the (r, Q) model under VMI with partial back-ordering imposes lower costs in comparison with (R, T) to the whole chain when parameters are fixed. Hence, the (r, Q) model under VMI is more rational for the aim of the cost-reduction of the total SC compared to the (R, T) model. Besides, (r, Q) is a better policy for systems where responsiveness plays a key role as (r, Q) imposes lower lost-sales and back-ordering on the whole chain compared to (R, T). Thus, implementing the proposed VMI with (r, Q) could be more cost-efficient and simultaneously more responsive. Conversely, for example, (R, T) is preferred when there is a low probability of facing lost-sales or where holding costs are very high for an inventory system. Therefore, the selection between (r, Q) and (R, T) must be made according to the context.

In this paper, different buyers with different cost parameters are assumed to be served with a common cycle time. Here, a common delivery cycle is used to ship items to two buyers. Developing the counterpart VMI system with different cycle times for buyers may be an exciting topic for future studies. Moreover, the lot-for-lot policy is used while a production run can provide many time deliveries for the buyers. The other researchers, as future works, can investigate those topics.

Acknowledgements

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Appendix A.

In the (*R*, *T*) VMI, we have:

$$\frac{\partial \overline{b}\left(R,l\left(DT\right)+T\right)}{\partial R}=\frac{\partial}{\partial R}\int_{R}^{\infty}(x-R)\,f(x)dx=-\int_{R}^{\infty}xf(x)dx=-\overline{F}(R) \quad \text{(A1)}$$

Also, for the (*r, Q*) VMI we have:

$$\frac{\partial \overline{b}\left(R,l\left(DT\right)\right)}{\partial r}=\frac{\partial}{\partial r}\int_{r}^{\infty}(x-r)\,f(x)dx=-\int_{r}^{\infty}xf(x)dx=-\overline{F}(r) \quad \text{(A2)}$$

Appendix B.

To calculate the $\overline{b}\left(R\right)$ in (*R, T*) replenishment review, it is assumed that demand during the lead time follows normal distribution with $x\sim N\left(D(l\left(DT\right)+T),\sigma^{2}\left(l\left(DT\right)+T\right)\right)$. Furthermore, the normal distribution function is as follows:

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}\sqrt{l(DT)+T}}e^{-1/2(\frac{x-R}{\sigma\sqrt{l(DT)+T}})^{2}} \quad \text{(B1)}$$

l(DT)+T is equal to $\frac{DT}{2P}+\max\{b_1,b_2\}+T$. Thus the average backorder for a buyer during the lead time will be calculated as follows:

$$\bar{b}\left(R,l\left(DT\right)+T\right)=\int_R^\infty\left(x-R\right)f\left(x\right)dx=\int_R^\infty\left(x-R\right)\frac{1}{\sqrt{2\pi}\sigma\sqrt{l\left(DT\right)+T}}e^{-\frac{1}{2}\left(\frac{x-R}{\sigma\sqrt{l\left(DT\right)+T}}\right)^2}dx \tag{B2}$$

The following procedure is presented for calculating the above equation:

$$\begin{aligned}&\frac{x-R}{\sigma\sqrt{l\left(DT\right)+T}}=u\rightarrow dx=\sigma\sqrt{l\left(DT\right)+T}\,du\\&\frac{R-D\left(l\left(DT\right)+T\right)}{\sigma\sqrt{l(DT)+T}}=k\rightarrow R=k\sigma\sqrt{l(DT)+T}+D\left(l\left(DT\right)+T\right)\end{aligned} \tag{B3}$$

$$\bar{b}\left(R,l\left(DT\right)+T\right)=\sigma\sqrt{l\left(DT\right)+T}\int_k^\infty\left(u-k\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u)^2}du=\sigma\sqrt{l\left(DT\right)+T}\,Gu(k)=\sigma\sqrt{DT/2P+\max\{b_1,b_2\}+T}\,L^{'}(u) \tag{B4}$$

$L^{'}(u)$ is the right-hand unit common linear loss integral.

Also, the procedure for calculating $\bar{b}\left(r,l\left(Q\right)\right)$ in (*r*, *Q*) system is similar to the above. It is worthy of mentioning that the parameter *u* in (*r*, *Q*) is equal to $u=\frac{x-r}{\sigma\sqrt{l\left(Q\right)}}$ should be calculated for finding the value of $L^{'}(u)$ .As well, note that $l\left(Q\right)=\frac{DT}{2P}+\max\left(b_1,b_2\right)$ for the proposed (*r*, *Q*) model.